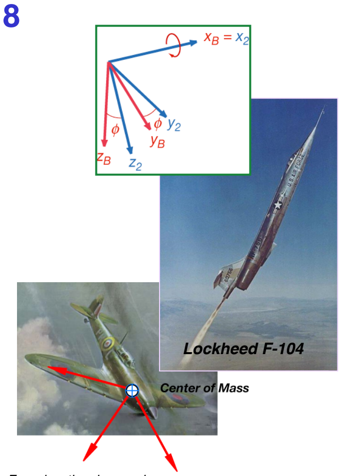
Aircraft Equations of Motion: Translation and Rotation

Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2018

Learning Objectives

- What use are the equations of motion?
- How is the angular orientation of the airplane described?
- What is a cross-product-equivalent matrix?
- What is angular momentum?
- How are the inertial properties of the airplane described?
- How is the rate of change of angular momentum calculated?



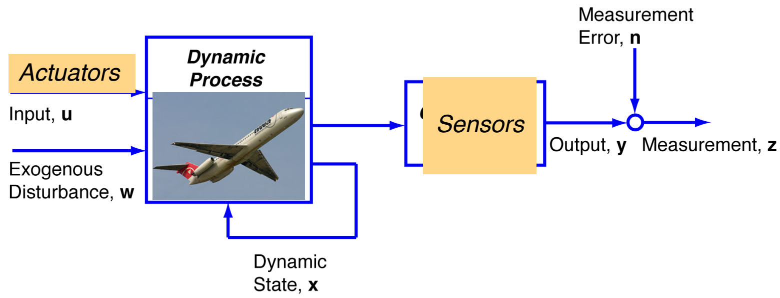
Reading:
Flight Dynamics
155-161

http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

Review Questions

- *What characteristic(s) provide maximum gliding range?*
- *Do gliding heavy airplanes fall out of the sky faster than light airplanes?*
- *Are the factors for maximum gliding range and minimum sink rate the same?*
- *How does the maximum climb rate vary with altitude?*
- *What are "energy height" and "specific excess power"?*
- *What is an "energy climb"?*
- *How is the "maneuvering envelope" defined?*
- *What factors determine the maximum steady turning rate?*

Dynamic Systems



Dynamic Process: Current state depends on prior state

- x = dynamic state
- u = input
- w = exogenous disturbance
- p = parameter
- *t* or *k* = time or event index

Observation Process: Measurement may contain error or be incomplete

- y = output (error-free)
- z = measurement
- n = measurement error

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t\right]$$

$$\mathbf{y}(t) = \mathbf{h}\left[\mathbf{x}(t),\mathbf{u}(t)\right]$$
$$\mathbf{z}(t) = \mathbf{y}(t) + \mathbf{n}(t)$$

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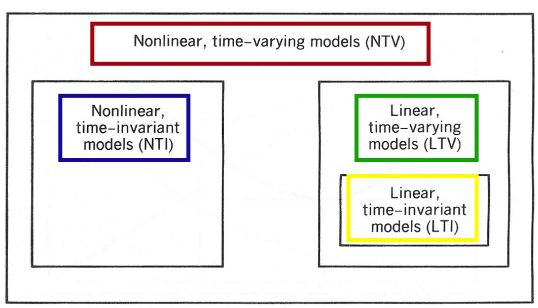
Ordinary Differential Equations Fall Into 4 Categories



$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t\right]$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t)$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t)\right]$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t) + \mathbf{L}\,\mathbf{w}(t)$$

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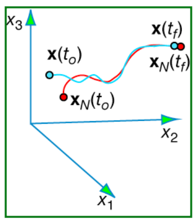
What Use are the Equations of Motion?

- **Nonlinear equations of motion**
  - **Compute "exact" flight paths and motions**
    - **Simulate flight motions**
    - **Optimize flight paths**
    - **Predict performance**
  - **Provide basis for approximate solutions**

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}\left[\mathbf{x}(t),\mathbf{u}(t),\mathbf{w}(t),\mathbf{p}(t),t\right]$$

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{G}\,\mathbf{u}(t) + \mathbf{L}\,\mathbf{w}(t)$$



- **Linear equations of motion**
  - **Simplify computation of flight paths and solutions**
  - **Define modes of motion**
  - **Provide basis for control system design and flying qualities analysis**

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Examples of Airplane Dynamic System Models

- **Nonlinear, Time-Varying**
  - **Large amplitude motions**
  - **Significant change in mass**

- **Nonlinear, Time-Invariant**
  - **Large amplitude motions**
  - **Negligible change in mass**

- **Linear, Time-Varying**
  - **Small amplitude motions**
  - **Perturbations from a dynamic flight path**

- **Linear, Time-Invariant**
  - **Small amplitude motions**
  - **Perturbations from an equilibrium flight path**

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Translational Position

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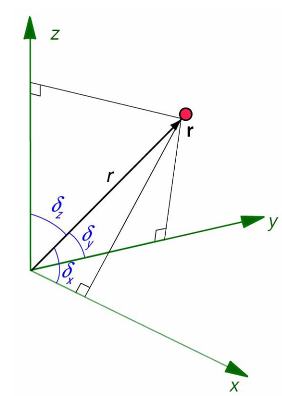
Position of a Particle

Projections of vector magnitude on three axes

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \cos\delta_x \\ \cos\delta_y \\ \cos\delta_z \end{bmatrix}$$

$$\begin{bmatrix} \cos\delta_x \\ \cos\delta_y \\ \cos\delta_z \end{bmatrix} = \textbf{Direction cosines}$$



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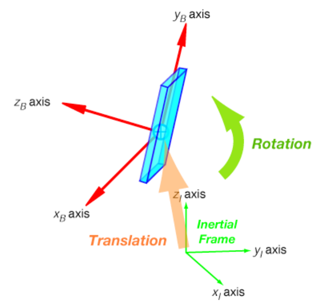
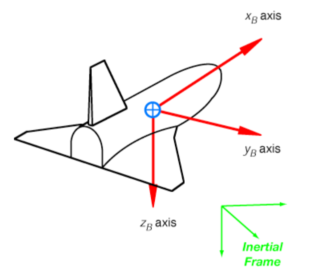
Cartesian Frames of Reference

- Two reference frames of interest
  - *I*: Inertial frame (fixed to inertial space)
  - *B*: Body frame (fixed to body)

- Translation
  - Relative linear positions of origins
- Rotation
  - Orientation of the body frame with respect to the inertial frame



Common convention (*z* up)



Aircraft convention (*z* down)

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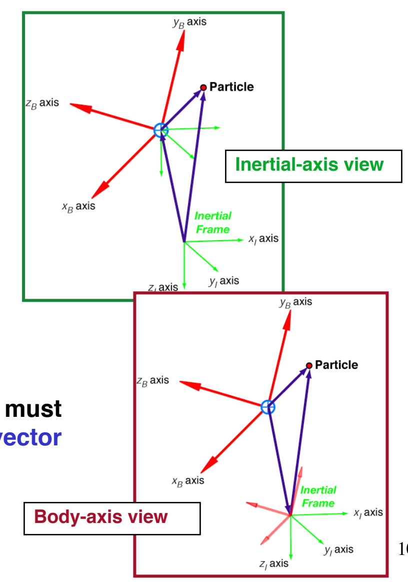
Measurement of Position in Alternative Frames - 1

- Two reference frames of interest
  - *I*: Inertial frame (fixed to inertial space)
  - *B*: Body frame (fixed to body)

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{r}_{particle} = \mathbf{r}_{origin} + \Delta\mathbf{r}_{w.r.t.origin}$$

- Differences in frame orientations must be taken into account in adding vector components



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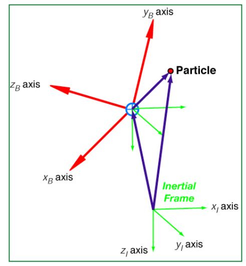
Measurement of Position in Alternative Frames - 2

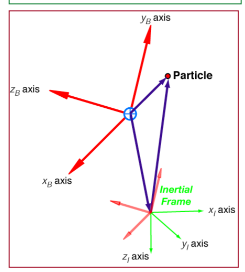
Inertial-axis view

$$\mathbf{r}_{particle_I} = \mathbf{r}_{origin-B_I} + \mathbf{H}_B^I \Delta\mathbf{r}_B$$



Body-axis view

$$\mathbf{r}_{particle_B} = \mathbf{r}_{origin-I_B} + \mathbf{H}_I^B \Delta\mathbf{r}_I$$



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Rotational Orientation

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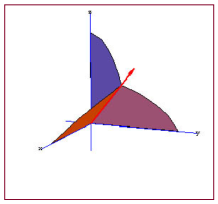
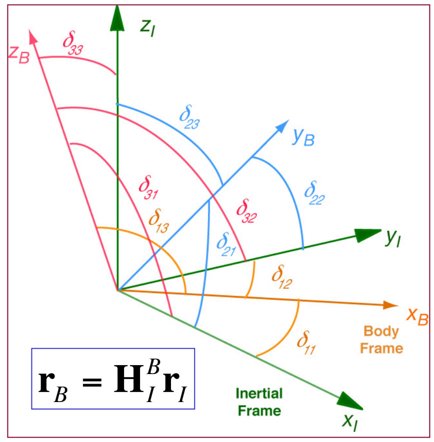


Direction Cosine Matrix

$$\mathbf{H}_I^B = \begin{bmatrix} \cos\delta_{11} & \cos\delta_{21} & \cos\delta_{31} \\ \cos\delta_{12} & \cos\delta_{22} & \cos\delta_{32} \\ \cos\delta_{13} & \cos\delta_{23} & \cos\delta_{33} \end{bmatrix}$$

- Projections of unit vector components of one reference frame on another
- Rotational orientation of one reference frame with respect to another
- Cosines of angles between each *I* axis and each *B* axis



$$\mathbf{r}_B = \mathbf{H}_I^B \mathbf{r}_I$$

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Properties of the Rotation Matrix

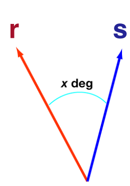
$$\mathbf{H}_I^B = \begin{bmatrix} \cos\delta_{11} & \cos\delta_{21} & \cos\delta_{31} \\ \cos\delta_{12} & \cos\delta_{22} & \cos\delta_{32} \\ \cos\delta_{13} & \cos\delta_{23} & \cos\delta_{33} \end{bmatrix}_I^B = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}_I^B$$

$$\mathbf{r}_B = \mathbf{H}_I^B \mathbf{r}_I \qquad \mathbf{s}_B = \mathbf{H}_I^B \mathbf{s}_I$$

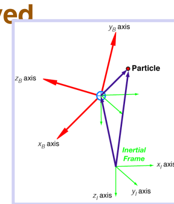
Orthonormal transformation

Angles between vectors are preserved

Lengths are preserved



$$|\mathbf{r}_I| = |\mathbf{r}_B| \quad ; \quad |\mathbf{s}_I| = |\mathbf{s}_B|$$
$$\angle(\mathbf{r}_I, \mathbf{s}_I) = \angle(\mathbf{r}_B, \mathbf{s}_B) = x \deg$$



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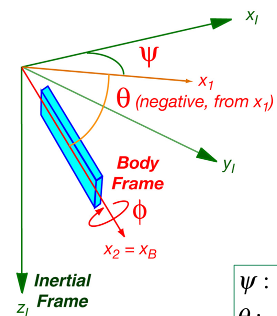
Euler Angles

- **Body attitude measured with respect to inertial frame**
- **Three-angle orientation expressed by sequence of three orthogonal single-angle rotations**

$$Inertial \Rightarrow Intermediate_1 \Rightarrow Intermediate_2 \Rightarrow Body$$



- **24 (±12) possible sequences of single-axis rotations**
- **Aircraft convention: 3-2-1, *z* positive down**

ψ : Yaw angle
θ : Pitch angle
ϕ : Roll angle

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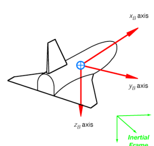
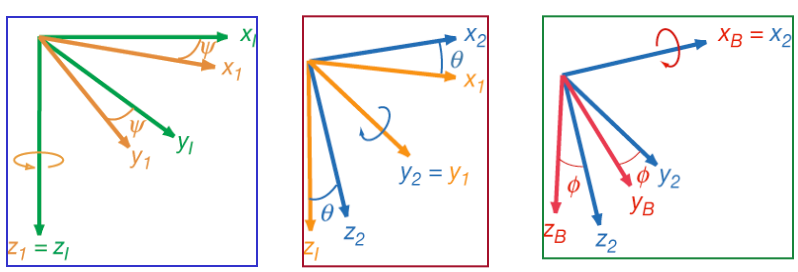


Euler Angles Measure the Orientation of One Frame with Respect to the Other

- **Conventional sequence of rotations from inertial to body frame**
  - **Each rotation is about a single axis**
  - **Right-hand rule**
  - **Yaw, then pitch, then roll**
  - **These are called Euler Angles**



Yaw rotation (ψ) about z_I **Pitch rotation (θ) about y_1** **Roll rotation (ϕ) about x_2**

Other sequences of 3 rotations can be chosen; however, once sequence is chosen, it must be retained

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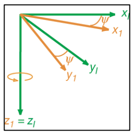
Reference Frame Rotation from Inertial to Body: Aircraft Convention (3-2-1)

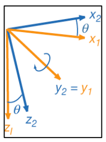
Yaw rotation (ψ) about z_I axis



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_1 = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_I = \begin{bmatrix} x_I \cos\psi + y_I \sin\psi \\ -x_I \sin\psi + y_I \cos\psi \\ z_I \end{bmatrix}$$

$$\mathbf{r}_1 = \mathbf{H}_I^1 \mathbf{r}_I$$

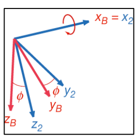
Pitch rotation (θ) about y_1 axis



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_2 = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_1$$

$$\mathbf{r}_2 = \mathbf{H}_1^2 \mathbf{r}_1 = \left[\mathbf{H}_1^2 \mathbf{H}_I^1\right] \mathbf{r}_I = \mathbf{H}_I^2 \mathbf{r}_I$$

Roll rotation (ϕ) about x_2 axis



$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_2$$

$$\mathbf{r}_B = \mathbf{H}_2^B \mathbf{r}_2 = \left[\mathbf{H}_2^B \mathbf{H}_1^2 \mathbf{H}_I^1\right] \mathbf{r}_I = \mathbf{H}_I^B \mathbf{r}_I$$

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The Rotation Matrix

The three-angle rotation matrix is the product of 3 single-angle rotation matrices:

$$\mathbf{H}_I^B(\phi,\theta,\psi) = \mathbf{H}_2^B(\phi)\mathbf{H}_1^2(\theta)\mathbf{H}_I^1(\psi)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\cos\phi\sin\psi + \sin\phi\sin\theta\cos\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \sin\phi\cos\theta \\ \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi & -\sin\phi\cos\psi + \cos\phi\sin\theta\sin\psi & \cos\phi\cos\theta \end{bmatrix}$$

an expression of the ***Direction Cosine Matrix***

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Rotation Matrix Inverse

Inverse relationship: interchange sub- and superscripts

$$\mathbf{r}_B = \mathbf{H}_I^B \mathbf{r}_I$$

$$\mathbf{r}_I = \left(\mathbf{H}_I^B\right)^{-1} \mathbf{r}_B = \mathbf{H}_B^I \mathbf{r}_B$$

Because transformation is orthonormal

Inverse = transpose

Rotation matrix is always non-singular

$$\left[\mathbf{H}_I^B(\phi,\theta,\psi)\right]^{-1} = \left[\mathbf{H}_I^B(\phi,\theta,\psi)\right]^T = \mathbf{H}_B^I(\psi,\theta,\phi)$$

$$\mathbf{H}_B^I = \left(\mathbf{H}_I^B\right)^{-1} = \left(\mathbf{H}_I^B\right)^T = \mathbf{H}_1^I \mathbf{H}_2^1 \mathbf{H}_B^2$$

$$\mathbf{H}_B^I \mathbf{H}_I^B = \mathbf{H}_I^B \mathbf{H}_B^I = \mathbf{I}$$

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Checklist

- ❑ *What are direction cosines?*
- ❑ *What are Euler angles?*
- ❑ *What rotation sequence is used to describe airplane attitude?*
- ❑ *What are properties of the rotation matrix?*

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Angular Momentum

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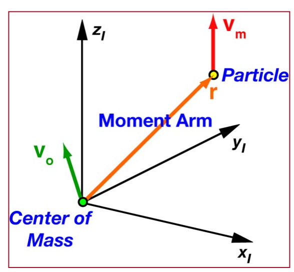
Angular Momentum of a Particle



- **Moment of linear momentum of differential particles that make up the body**
  - **(Differential masses) x components of the velocity that are perpendicular to the moment arms**

$$d\mathbf{h} = (\mathbf{r} \times dm\ \mathbf{v}) = (\mathbf{r} \times \mathbf{v}_m)\,dm$$
$$= [\mathbf{r} \times (\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r})]\,dm$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

- **Cross Product: Evaluation of a determinant with unit vectors (*i, j, k*) along axes, (*x, y, z*) and (v_x, v_y, v_z) projections on to axes**

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = (yv_z - zv_y)\boldsymbol{i} + (zv_x - xv_z)\boldsymbol{j} + (xv_y - yv_x)\boldsymbol{k}$$

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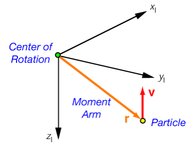

Cross-Product-Equivalent Matrix

Cross product

$$\mathbf{r} \times \mathbf{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = \left(yv_z - zv_y\right)\boldsymbol{i} + \left(zv_x - xv_z\right)\boldsymbol{j} + \left(xv_y - yv_x\right)\boldsymbol{k}$$

$$= \begin{bmatrix} \left(yv_z - zv_y\right) \\ \left(zv_x - xv_z\right) \\ \left(xv_y - yv_x\right) \end{bmatrix} = \tilde{\mathbf{r}}\mathbf{v} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Cross-product-equivalent matrix

$$\tilde{\mathbf{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

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Angular Momentum of the Aircraft

- Integrate moment of linear momentum of differential particles over the body

$$\mathbf{h} = \int_{Body} \left[\mathbf{r} \times \left(\mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r}\right)\right] dm = \int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} \int_{z_{min}}^{z_{max}} \left(\mathbf{r} \times \mathbf{v}\right) \rho(x,y,z)\, dx\, dy\, dz = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix}$$

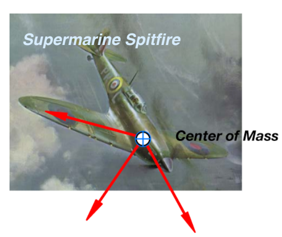
$\rho(x,y,z) =$ Density of the body

- Choose the center of mass as the rotational center

$$\mathbf{h} = \int_{Body} \left(\mathbf{r} \times \mathbf{v}_o\right) dm + \int_{Body} \left[\mathbf{r} \times \left(\boldsymbol{\omega} \times \mathbf{r}\right)\right] dm$$

$$= 0 - \int_{Body} \left[\mathbf{r} \times \left(\mathbf{r} \times \boldsymbol{\omega}\right)\right] dm$$

$$= -\int_{Body} \left(\mathbf{r} \times \mathbf{r}\right) dm \times \boldsymbol{\omega} \equiv -\int_{Body} \left(\tilde{\mathbf{r}}\tilde{\mathbf{r}}\right) dm\, \boldsymbol{\omega}$$



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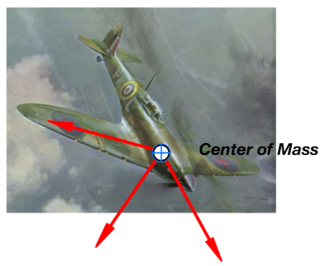
Location of the Center of Mass

$$\mathbf{r}_{cm} = \frac{1}{m}\int_{Body} \mathbf{r}\,dm = \frac{1}{m}\int_{x_{\min}}^{x_{\max}}\int_{y_{\min}}^{y_{\max}}\int_{z_{\min}}^{z_{\max}} \mathbf{r}\rho(x,y,z)\,dx\,dy\,dz = \begin{bmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \end{bmatrix}$$



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The Inertia Matrix

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The Inertia Matrix

$$\mathbf{h} = -\int_{Body} \tilde{\mathbf{r}}\tilde{\mathbf{r}}\,\boldsymbol{\omega}\,dm = -\int_{Body} \tilde{\mathbf{r}}\tilde{\mathbf{r}}\,dm\,\boldsymbol{\omega} = \mathbb{I}\boldsymbol{\omega}$$

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where

$$\mathbb{I} = -\int_{Body} \tilde{\mathbf{r}}\tilde{\mathbf{r}}\,dm = -\int_{Body} \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}\begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} dm$$

$$= \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm$$

Inertia matrix derives from equal effect of angular rate on all particles of the aircraft

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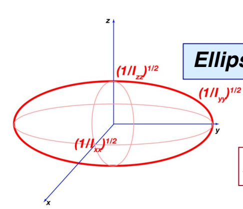
Moments and Products of Inertia

$$\mathbb{I} = \int_{Body} \begin{bmatrix} (y^2+z^2) & -xy & -xz \\ -xy & (x^2+z^2) & -yz \\ -xz & -yz & (x^2+y^2) \end{bmatrix} dm = \begin{bmatrix} \mathbb{I}_{xx} & -\mathbb{I}_{xy} & -\mathbb{I}_{xz} \\ -\mathbb{I}_{xy} & \mathbb{I}_{yy} & -\mathbb{I}_{yz} \\ -\mathbb{I}_{xz} & -\mathbb{I}_{yz} & \mathbb{I}_{zz} \end{bmatrix}$$

Inertia matrix

- **Moments of inertia on the diagonal**
- **Products of inertia off the diagonal**
- **If products of inertia are zero, (x, y, z) are principal axes --->**
- **All rigid bodies have a set of principal axes**

$$\begin{bmatrix} \mathbb{I}_{xx} & 0 & 0 \\ 0 & \mathbb{I}_{yy} & 0 \\ 0 & 0 & \mathbb{I}_{zz} \end{bmatrix}$$



Ellipsoid of Inertia

$$\mathbb{I}_{xx}x^2 + \mathbb{I}_{yy}y^2 + \mathbb{I}_{zz}z^2 = 1$$

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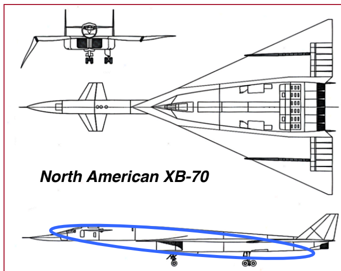
Inertia Matrix of an Aircraft with Mirror Symmetry

$$\mathbb{I} = \int_{Body} \begin{bmatrix} (y^2+z^2) & 0 & -xz \\ 0 & (x^2+z^2) & 0 \\ -xz & 0 & (x^2+y^2) \end{bmatrix} dm = \begin{bmatrix} \mathbb{I}_{xx} & 0 & -\mathbb{I}_{xz} \\ 0 & \mathbb{I}_{yy} & 0 \\ -\mathbb{I}_{xz} & 0 & \mathbb{I}_{zz} \end{bmatrix}$$

Nose high/low product of inertia, I_{xz}



Nominal Configuration

Tips folded, 50% fuel, W = 38,524 lb

x_{cm} @0.218 $\bar{c}$

$\mathbb{I}_{xx} = 1.8 \times 10^6$ slug-ft^2

$\mathbb{I}_{yy} = 19.9 \times 10^6$ slug-ft^2

$\mathbb{I}_{xx} = 22.1 \times 10^6$ slug-ft^2

$\mathbb{I}_{xz} = -0.88 \times 10^6$ slug-ft^2

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Checklist

- ❑ How is the location of the center of mass found?
- ❑ What is a cross-product-equivalent matrix?
- ❑ What is the inertia matrix?
- ❑ What is an ellipsoid of inertia?
- ❑ What does the "nose-high" product of inertia represent?

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Historical Factoids

Technology of World War II Aviation

- **1938-45:** Analytical and experimental approach to design
  - Many configurations designed and flight-tested
  - Increased specialization; radar, navigation, and communication
  - Approaching the "sonic barrier"
- **Aircraft Design**
  - Large, powerful, high-flying aircraft
  - Turbocharged engines
  - Oxygen and Pressurization

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Power Effects on Stability and Control

- ***Brewster Buffalo***: over-armored and under-powered
- During W.W.II, the size of fighters remained about the same, but installed horsepower doubled (*F4F* vs. *F8F*)
- Use of flaps means high power at low speed, increasing relative significance of thrust effects

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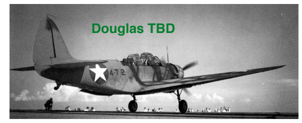
World War II Carrier-Based Airplanes

- Takeoff **without catapult**, relatively low landing speed
  http://www.youtube.com/watch?v=4dySbhK1vNk
- Tailhook and arresting gear
- Carrier steams into wind
- **Design for storage** (short tail length, folding wings) affects stability and control



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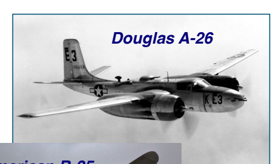
Multi-Engine Aircraft of World War II

- Large W.W.II aircraft had unpowered controls:
  - High foot-pedal force
  - Rudder stability problems arising from balancing to reduce pedal force
- Severe engine-out problem for **twin-engine aircraft**



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WW II Military Flying Boats

Seaplanes proved useful during World War II

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Rate of Change of Angular Momentum

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Newton's 2nd Law, Applied to Rotational Motion

In inertial frame, rate of change of angular momentum = applied moment (or torque), M

$$\frac{d\mathbf{h}}{dt} = \frac{d(\mathbb{I}\boldsymbol{\omega})}{dt} = \frac{d\mathbb{I}}{dt}\boldsymbol{\omega} + \mathbb{I}\frac{d\boldsymbol{\omega}}{dt}$$

$$= \dot{\mathbb{I}}\boldsymbol{\omega} + \mathbb{I}\dot{\boldsymbol{\omega}} = \mathbf{M} = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}$$

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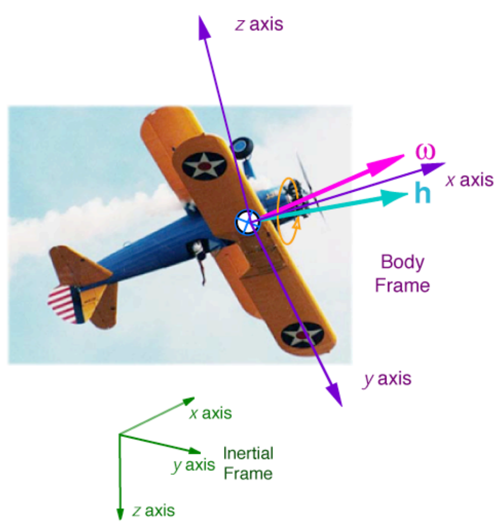
Angular Momentum and Rate

Angular momentum and rate vectors are not necessarily aligned

$$\mathbf{h} = \mathbb{I}\boldsymbol{\omega}$$



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How Do We Get Rid of d𝕀/dt in the Angular Momentum Equation?

Chain Rule

$$\frac{d(\mathbb{I}\boldsymbol{\omega})}{dt} = \dot{\mathbb{I}}\boldsymbol{\omega} + \mathbb{I}\dot{\boldsymbol{\omega}}$$

... and in an inertial frame

$$\dot{\mathbb{I}} \neq 0$$

- **Dynamic equation in a body-referenced frame**
  - **Inertial properties of a constant-mass, rigid body are unchanging in a body frame of reference**
  - **... but a body-referenced frame is "non-Newtonian" or "non-inertial"**
  - **Therefore, dynamic equation must be modified for expression in a rotating frame**

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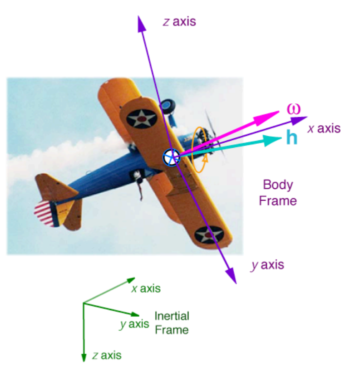
Angular Momentum Expressed in Two Frames of Reference



- **Angular momentum and rate are vectors**
  - **Expressed in either the inertial or body frame**
  - **Two frames related algebraically by the rotation matrix**

$$\mathbf{h}_B(t) = \mathbf{H}_I^B(t)\mathbf{h}_I(t); \qquad \mathbf{h}_I(t) = \mathbf{H}_B^I(t)\mathbf{h}_B(t)$$

$$\boldsymbol{\omega}_B(t) = \mathbf{H}_I^B(t)\boldsymbol{\omega}_I(t); \qquad \boldsymbol{\omega}_I(t) = \mathbf{H}_B^I(t)\boldsymbol{\omega}_B(t)$$

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Vector Derivative Expressed in a Rotating Frame

Chain Rule

Effect of body-frame rotation

$$\dot{\mathbf{h}}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \dot{\mathbf{H}}_B^I \mathbf{h}_B$$

Rate of change expressed in body frame

Alternatively

$$\dot{\mathbf{h}}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \boldsymbol{\omega}_I \times \mathbf{h}_I = \mathbf{H}_B^I \dot{\mathbf{h}}_B + \tilde{\boldsymbol{\omega}}_I \mathbf{h}_I$$

Consequently, the 2nd term is

$$\dot{\mathbf{H}}_B^I \mathbf{h}_B = \tilde{\boldsymbol{\omega}}_I \mathbf{h}_I = \tilde{\boldsymbol{\omega}}_I \mathbf{H}_B^I \mathbf{h}_B$$

... where the cross-product equivalent matrix of angular rate is

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

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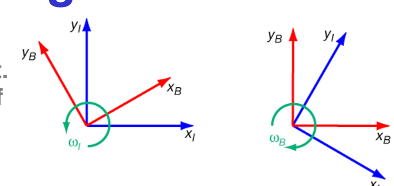
External Moment Causes Change in Angular Rate

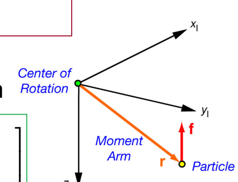
Positive rotation of Frame B w.r.t. Frame A is a **negative** rotation of Frame A w.r.t. Frame B



In the body frame of reference, the angular momentum change is

$$\begin{aligned}\dot{\mathbf{h}}_B &= \mathbf{H}_I^B \dot{\mathbf{h}}_I + \dot{\mathbf{H}}_I^B \mathbf{h}_I = \mathbf{H}_I^B \dot{\mathbf{h}}_I - \boldsymbol{\omega}_B \times h_B = \mathbf{H}_I^B \dot{\mathbf{h}}_I - \tilde{\boldsymbol{\omega}}_B h_B \\ &= \mathbf{H}_I^B \mathbf{M}_I - \tilde{\boldsymbol{\omega}}_B \mathbb{I}_B \boldsymbol{\omega}_B = \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B \mathbb{I}_B \boldsymbol{\omega}_B\end{aligned}$$

Moment = torque = force x moment arm



$$\mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}_I ; \quad \mathbf{M}_B = \mathbf{H}_I^B \mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}_B = \begin{bmatrix} L \\ M \\ N \end{bmatrix}$$

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Rate of Change of Body-Referenced Angular Rate due to External Moment

In the body frame of reference, the angular momentum change is

$$\begin{aligned}\dot{\mathbf{h}}_B &= \mathbf{H}_I^B\dot{\mathbf{h}}_I + \dot{\mathbf{H}}_I^B\mathbf{h}_I = \mathbf{H}_I^B\dot{\mathbf{h}}_I - \boldsymbol{\omega}_B \times h_B \\ &= \mathbf{H}_I^B\dot{\mathbf{h}}_I - \tilde{\boldsymbol{\omega}}_B h_B = \mathbf{H}_I^B\mathbf{M}_I - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B \\ &= \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B\end{aligned}$$

For constant body-axis inertia matrix

$$\dot{\mathbf{h}}_B = \mathbb{I}_B\dot{\boldsymbol{\omega}}_B = \mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B$$

Consequently, the differential equation for angular rate of change is

$$\dot{\boldsymbol{\omega}}_B = \mathbb{I}_B^{-1}\left(\mathbf{M}_B - \tilde{\boldsymbol{\omega}}_B\mathbb{I}_B\boldsymbol{\omega}_B\right)$$

Checklist

- ❑ Why is it inconvenient to solve momentum rate equations in an inertial reference frame?
- ❑ Are angular rate and momentum vectors aligned?
- ❑ How are angular rate equations transformed from an inertial to a body frame?

Next Time:
Aircraft Equations of Motion: Flight Path Computation

Reading:
Flight Dynamics
161-180

Learning Objectives

How is a rotating reference frame described in an inertial reference frame?
Is the transformation singular?
What adjustments must be made to expressions for forces and moments in a non-inertial frame?
How are the 6-DOF equations implemented in a computer?
Damping effects

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SUPPLEMENTAL MATERIAL

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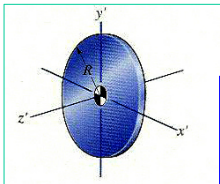
Moments and Products of Inertia

(Bedford & Fowler)

Moments and products of inertia tabulated for geometric shapes with uniform density

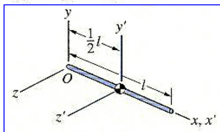
Construct aircraft moments and products of inertia from components using parallel-axis theorem

Model in *CREO,* etc.



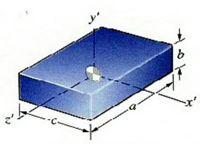
Thin circular plate

$$I_{x'\,\text{axis}} = I_{y'\,\text{axis}} = \frac{1}{4}mR^2, \qquad I_{z'\,\text{axis}} = \frac{1}{2}mR^2,$$
$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



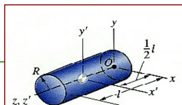
Slender bar

$$I_{x\,\text{axis}} = 0, \qquad I_{y\,\text{axis}} = I_{z\,\text{axis}} = \frac{1}{3}ml^2,$$
$$I_{xy} = I_{yz} = I_{zx} = 0.$$
$$I_{x'\,\text{axis}} = 0, \qquad I_{y'\,\text{axis}} = I_{z'\,\text{axis}} = \frac{1}{12}ml^2,$$
$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



Rectangular prism

Volume $= abc$
$$I_{x'\,\text{axis}} = \frac{1}{12}m(a^2 + b^2), \qquad I_{y'\,\text{axis}} = \frac{1}{12}m(a^2 + c^2),$$
$$I_{z'\,\text{axis}} = \frac{1}{12}m(b^2 + c^2), \qquad I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$



Circular cylinder

Volume $= \pi R^2 l$
$$I_{x\,\text{axis}} = I_{y\,\text{axis}} = m\left(\frac{1}{3}l^2 + \frac{1}{4}R^2\right), \qquad I_{z\,\text{axis}} - \frac{1}{2}mR^2,$$
$$I_{xy} = I_{yz} = I_{zx} = 0.$$
$$I_{x'\,\text{axis}} = I_{y'\,\text{axis}} = m\left(\frac{1}{12}l^2 + \frac{1}{4}R^2\right), \qquad I_{z'\,\text{axis}} = \frac{1}{2}mR^2,$$
$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0.$$

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