Root Locus Analysis of Parameter Variations and Feedback Control

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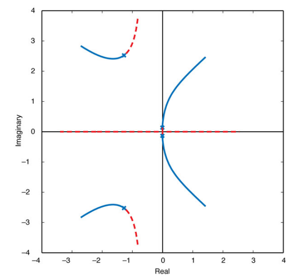
MAE 331, 2018

Learning Objectives

- **Effects of system parameter variations on modes of motion (expressed in characteristic equation)**
- **Root locus analysis**
  - **Evans's rules for construction**
  - **Application to longitudinal dynamic models**



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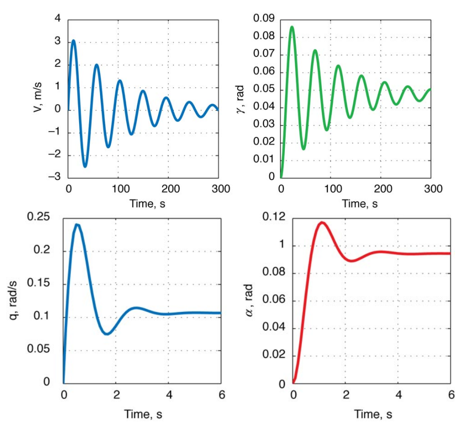
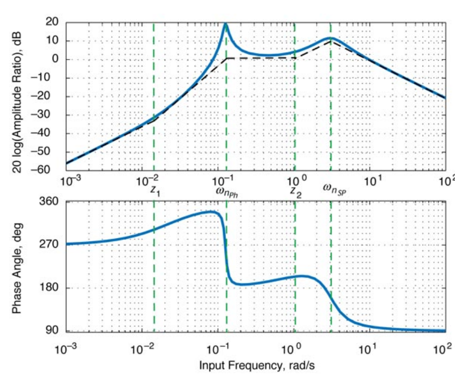
Aircraft Modes of Motion





How do the modes change as dynamic parameters change?

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Relationship of $(s\mathbf{I} - \mathbf{F})^{-1}$ to State Transition Matrix, $\Phi(t,0)$

Initial condition response

Time Domain

$$\Delta\mathbf{x}(t) = \mathbf{\Phi}(t,0)\Delta\mathbf{x}(0)$$

Frequency Domain

$$\Delta\mathbf{x}(s) = [s\mathbf{I} - \mathbf{F}]^{-1}\Delta\mathbf{x}(0)$$

$\Delta\mathbf{x}(s)$ is the Laplace transform of $\Delta\mathbf{x}(t)$

$$\Delta\mathbf{x}(s) = \left[(s\mathbf{I} - \mathbf{F})^{-1}\right]\Delta\mathbf{x}(0) = \mathcal{L}[\Delta\mathbf{x}(t)]$$
$$= \mathcal{L}\left[\mathbf{\Phi}(t,0)\Delta\mathbf{x}(0)\right] = \mathcal{L}\left[\mathbf{\Phi}(t,0)\right]\Delta\mathbf{x}(0)$$

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Relationship of $(s\mathbf{I} - \mathbf{F})^{-1}$ to State Transition Matrix, $\Phi(t,0)$

Therefore,

$$[s\mathbf{I} - \mathbf{F}]^{-1} = \mathcal{L}\left[\mathbf{\Phi}(t,0)\right]$$
$$= \text{Laplace transform of the state transition matrix}$$

Alternative way to calculate the state transition matrix,

$$\mathbf{\Phi}(t,0) = \mathcal{L}^{-1}[s\mathbf{I} - \mathbf{F}]^{-1}$$

In time or frequency domain, modes of motion depend on F

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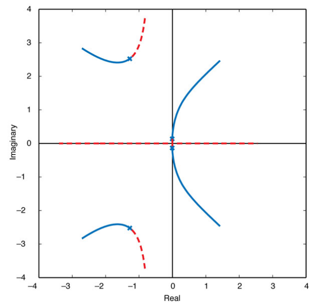
How do the roots vary?

Characteristic equation defines the modes of motion

$$\left|s\mathbf{I}-\mathbf{F}\right| = \Delta(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$
$$= (s-\lambda_1)(s-\lambda_2)(\ldots)(s-\lambda_n) = 0$$



- **Roots (eigenvalues) vary if a_i vary**
- **a_i are real, roots are real or complex**

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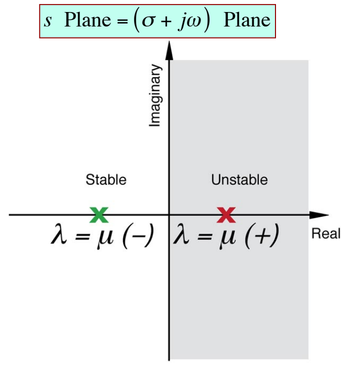
Real Roots

- **On real axis in *s* Plane**
- **Represents convergent or divergent time response**
- **Time constant, $\tau = -1/\lambda = -1/\mu$, sec**

$$\lambda_i = \mu_i \quad \text{(Real number)}$$
$$x(t) = x(0)e^{\mu t}$$



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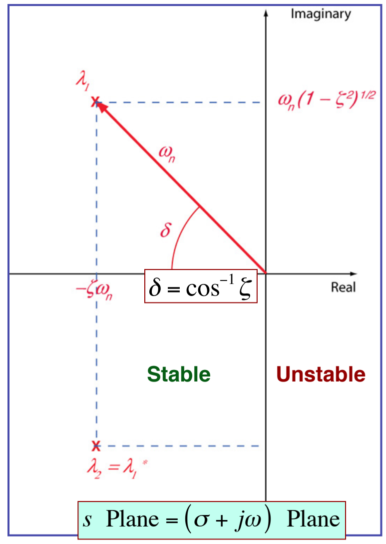
Complex Roots

- Occur only in complex-conjugate pairs
- Represent oscillatory modes
- Natural frequency, ω_n, and damping ratio, ζ, as shown

$$\lambda_1 = \mu_1 + j\nu_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}$$

$$\lambda_2 = \mu_2 + j\nu_2 = \mu_1 - j\nu_1 \triangleq \lambda_1^* = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$$



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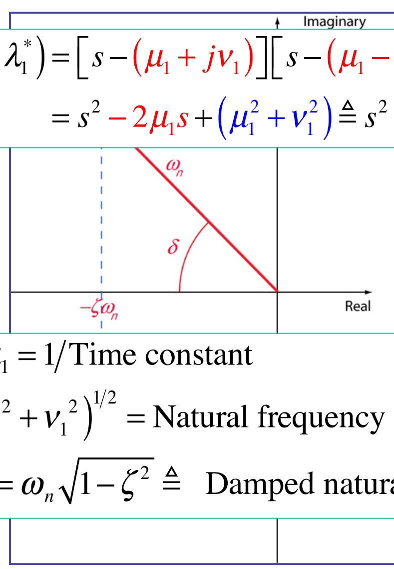
Natural Frequency, Damping Ratio, and Damped Natural Frequency

$$(s-\lambda_1)(s-\lambda_1^*) = \left[s-(\mu_1 + j\nu_1)\right]\left[s-(\mu_1 - j\nu_1)\right]$$
$$= s^2 - 2\mu_1 s + (\mu_1^2 + \nu_1^2) \triangleq s^2 + 2\zeta\omega_n s + \omega_n^2$$



$$\zeta\omega_n = -\mu_1 = 1/\text{Time constant}$$

$$\omega_n = (\mu_1^2 + \nu_1^2)^{1/2} = \text{Natural frequency}$$

$$\omega_{n_{damped}} = \nu_1 = \omega_n\sqrt{1-\zeta^2} \triangleq \text{Damped natural frequency}$$

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Longitudinal Characteristic Equation

$$\Delta_{Lon}(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

$$= s^4 + \left(D_V + L_\alpha/V_N - M_q\right)s^3$$

$$+\left[(g - D_\alpha)L_V/V_N + D_V\left(L_\alpha/V_N - M_q\right) - M_q L_\alpha/V_N - M_\alpha\right]s^2$$

$$+\left\{M_q\left[(D_\alpha - g)L_V/V_N - D_V L_\alpha/V_N\right] + D_\alpha M_V - D_V M_\alpha\right\}s$$

$$+ g\left(M_V L_\alpha/V_N - M_\alpha L_V/V_N\right) = 0$$

with $L_q = D_q = 0$

How do the roots vary when we change M_α?

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… or pitch-rate damping, M_q?

$$\Delta_{Lon}(s) = s^4 + \left(D_V + L_\alpha/V_N - M_q\right)s^3$$

$$+\left[(g - D_\alpha)L_V/V_N + D_V\left(L_\alpha/V_N - M_q\right) - M_q L_\alpha/V_N - M_\alpha\right]s^2$$

$$+\left\{M_q\left[(D_\alpha - g)L_V/V_N - D_V L_\alpha/V_N\right] + D_\alpha M_V - D_V M_\alpha\right\}s$$

$$+ g\left(M_V L_\alpha/V_N - M_\alpha L_V/V_N\right) = 0$$

TBD

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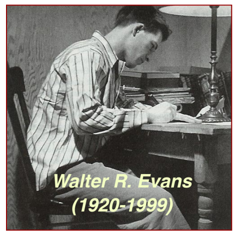
Evans's Rules for Root Locus Analysis



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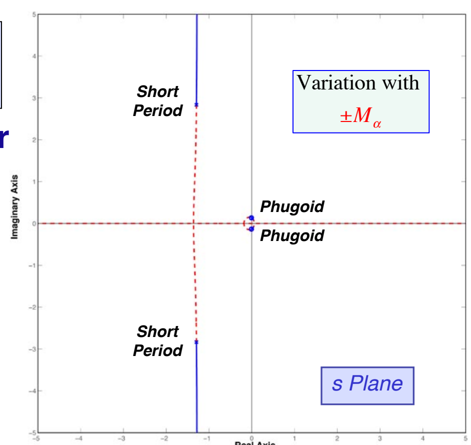
Root Locus Analysis of Parametric Effects on Aircraft Dynamics

Locus: "the set of all points whose location is determined by stated conditions"

- **Parametric variations alter eigenvalues of F**
- **With computers (e.g., MATLAB), repeatedly evaluate *eig.m***
- **Without .... ?**
- **Graphical technique for finding the roots as a parameter ("gain") varies**



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Parameters of Characteristic Polynomial

$$\Delta_{Lon}(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$
$$= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4)$$
$$= \left(s^2 + 2\zeta_P \omega_{n_P} s + \omega_{n_P}^2\right)\left(s^2 + 2\zeta_{SP}\omega_{n_{SP}} s + \omega_{n_{SP}}^2\right)$$

Break the polynomial into 2 parts to examine the effect of a single parameter

$$\Delta(s) = d(s) + k\, n(s) = 0$$

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Effect of a_0 Variation on Longitudinal Root Location

Example: Root locus gain, $k = a_0$

$$\Delta_{Lon}(s) = \left[s^4 + a_3 s^3 + a_2 s^2 + a_1 s\right] + [k] \triangleq d(s) + kn(s)$$
$$= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) = 0$$

$d(s)$: Polynomial in s, degree = n; there are n (= 4) poles

$n(s)$: Polynomial in s, degree = q; there are q (= 0) zeros

$$d(s) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s$$
$$= [s-(0)](s-\lambda'_2)(s-\lambda'_3)(s-\lambda'_4);\ \text{four poles, one at } 0$$
$$n(s) = 1;\ \text{no zeros}$$

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Effect of a_1 Variation on Longitudinal Root Location

Example: Root locus gain, $k = a_1$

$$\Delta_{Lon}(s) = s^4 + a_3 s^3 + a_2 s^2 + ks + a_0 \triangleq d(s) + kn(s)$$
$$= (s-\lambda_1)(s-\lambda_2)(s-\lambda_3)(s-\lambda_4) = 0$$

$d(s)$: Polynomial in s, degree = n; there are n (= 4) poles
$n(s)$: Polynomial in s, degree = q; there are q (= 1) zeros

$$d(s) = s^4 + a_3 s^3 + a_2 s^2 + a_0$$
$$= (s-\lambda'_1)(s-\lambda'_2)(s-\lambda'_3)(s-\lambda'_4); \text{ four poles}$$
$n(s) = s$; one zero at 0

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Three Equivalent Equations for Evaluating Locations of Roots

$$\Delta(s) = d(s) + k\,n(s) = 0$$

$$1 + k\frac{n(s)}{d(s)} = 0$$

$$k\frac{n(s)}{d(s)} = -1 = (1)e^{\pm j\pi(rad)} = (1)e^{\pm j180°}$$

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Evan's Root Locus Criterion

All points on the locus of roots must satisfy the equation

$$k\frac{n(s)}{d(s)} = -1$$

i.e., all points on <u>root locus</u> must have phase angle = ±180˚

$$k\frac{n(s)}{d(s)} = (1)e^{\pm j180°}$$

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Longitudinal Equation Example

Typical flight condition

$$\begin{aligned}\Delta_{Lon}(s) &= s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \\ &= s^4 + 2.57s^3 + 9.68s^2 + 0.202s + 0.145 \\ &= \left[s^2 + 2(0.0678)0.124s + (0.124)^2\right]\left[s^2 + 2(0.411)3.1s + (3.1)^2\right] = 0\end{aligned}$$

Phugoid | Short Period

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Effect of a_0 Variation

$$\Delta_{Lon}(s) = s^4 + 2.57s^3 + 9.68s^2 + 0.202s + 0.145 = 0$$

$k = a_0$

$$\begin{aligned}\Delta(s) &= s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \\ &= s\left(s^3 + a_3 s^2 + a_2 s + a_1\right) + k \\ &= s\left(s + 0.21\right)\left[s^2 + 2.55s + 9.62\right] + k\end{aligned}$$

Rearrange

$$\frac{k}{s\left(s+0.21\right)\left[s^2 + 2.55s + 9.62\right]} = -1$$

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Effect of a_1 Variation

$k = a_1$

$$\begin{aligned}\Delta(s) &= s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \\ &= s^4 + a_3 s^3 + a_2 s^2 + ks + a_0 \\ &= \left(s^4 + a_3 s^3 + a_2 s^2 + a_0\right) + ks \\ &= \left[s^2 - 0.00041s + 0.015\right]\left[s^2 + 2.57s + 9.67\right] + ks\end{aligned}$$

Rearrange

$$\frac{ks}{\left[s^2 - 0.00041s + 0.015\right]\left[s^2 + 2.57s + 9.67\right]} = -1$$

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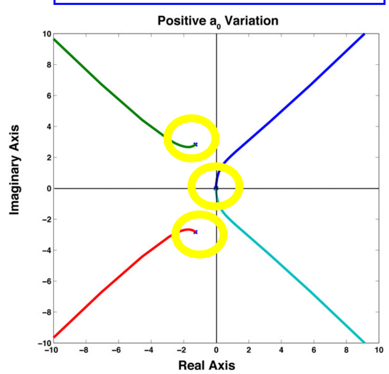
Origins of Roots ($k = 0$)

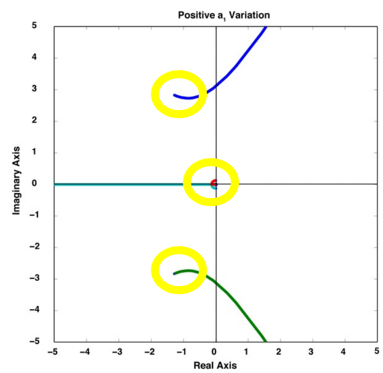
n poles of $d(s)$

$$\Delta(s) = d(s) + kn(s) \xrightarrow[k \to 0]{} d(s)$$

$$\frac{k}{s(s+0.21)\left[s^2 + 2.55s + 9.62\right]} = -1$$



$$\frac{ks}{\left[s^2 - 0.00041s + 0.015\right]\left[s^2 + 2.57s + 9.67\right]} = -1$$



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Destinations of Roots as k Becomes Large

1) q roots go to the zeros of $n(s)$

$$\frac{d(s) + kn(s)}{k} = \frac{d(s)}{k} + n(s) \xrightarrow[k \to \pm\infty]{} n(s) = (s - z_1)(s - z_2)\cdots$$

2) $(n - q)$ roots go to infinity

$$\left[\frac{d(s) + kn(s)}{n(s)}\right] = \left[\frac{d(s)}{n(s)} + k\right] \xrightarrow[k \to \pm R \to \pm\infty]{} \left[\frac{s^n}{s^q} + k\right] \to s^{(n-q)} \pm R \to \pm\infty$$

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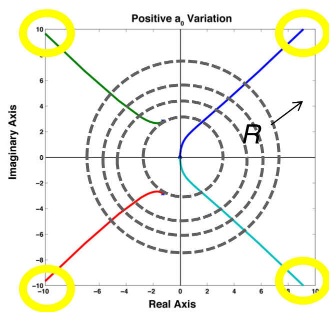
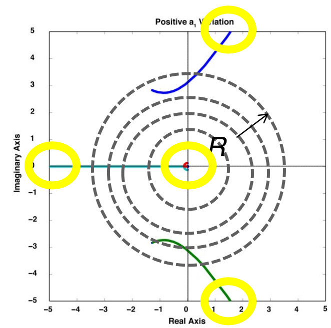
Destinations of Roots for Large *k*

$$\frac{k}{s(s+0.21)\left[s^2+2.55s+9.62\right]}=-1$$

$$\frac{ks}{\left[s^2-0.00041s+0.015\right]\left[s^2+2.57s+9.67\right]}=-1$$





(*n* – *q*) Roots Approach Asymptotes as *k* –> ±∞

Asymptote angles for positive *k*

$$\theta(rad)=\frac{\pi+2m\pi}{n-q}, \quad m=0,1,\ldots,(n-q)-1$$

Asymptote angles for negative *k*

$$\theta(rad)=\frac{2m\pi}{n-q}, \quad m=0,1,\ldots,(n-q)-1$$

Origin of Asymptotes = "Center of Gravity"

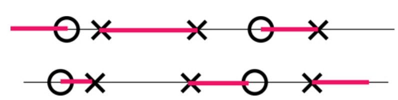
(Sum of real parts of poles minus sum of real parts of zeros)

(# of poles minus # of zeros)

$$"c.g." = \frac{\sum_{i=1}^{n} \sigma_{\lambda_i} - \sum_{j=1}^{q} \sigma_{z_j}}{n-q}$$

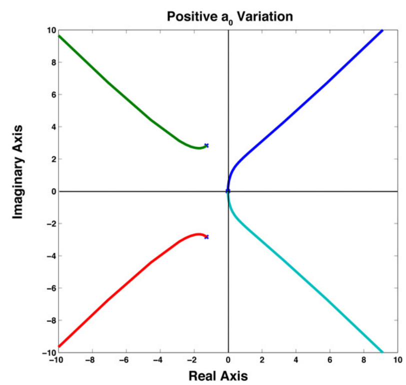
Root Locus on Real Axis

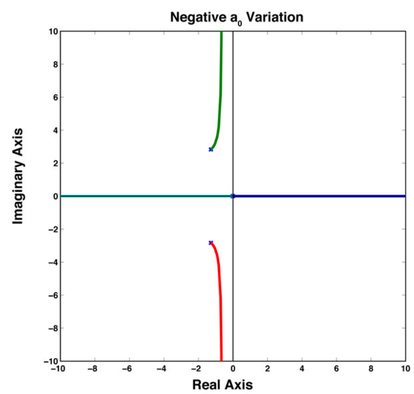
- **Locus on real axis**
  - **k > 0: Any segment with odd number of poles and zeros to the right on the axis**
  - **k < 0: Any segment with even number of poles and zeros to the right on the axis**



Roots for Positive and Negative Variations of $k = a_0$

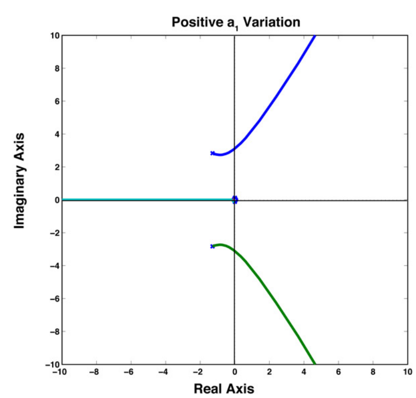
$$\frac{k}{s(s+0.21)\left[s^2+2.55s+9.62\right]} = -1$$

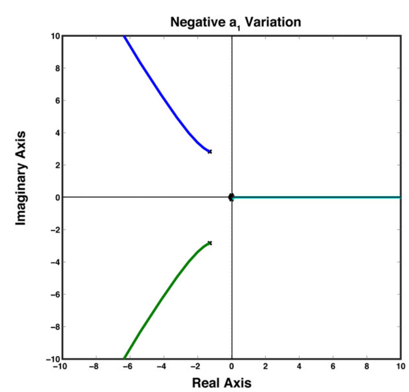




Roots for Positive and Negative Variations of $k = a_1$

$$\frac{ks}{\left[s^2-0.00041s+0.015\right]\left[s^2+2.57s+9.67\right]} = -1$$





Root Locus Analysis of Simplified Longitudinal Modes

Approximate Phugoid Model

2nd-order equation

$$\Delta \dot{\mathbf{x}}_{Ph} = \begin{bmatrix} \Delta \dot{V} \\ \Delta \dot{\gamma} \end{bmatrix} \approx \begin{bmatrix} -D_V & -g \\ L_V / V_N & 0 \end{bmatrix} \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix} + \begin{bmatrix} T_{\delta T} \\ L_{\delta T} / V_N \end{bmatrix} \Delta \delta T$$

Characteristic polynomial

$$\left| s\mathbf{I} - \mathbf{F}_{Ph} \right| = \det\left(s\mathbf{I} - \mathbf{F}_{Ph} \right) \equiv$$
$$\Delta(s) = s^2 + D_V s + g L_V / V_N$$
$$= s^2 + 2\zeta \omega_n s + \omega_n{}^2$$

$$\omega_{n_P} = \sqrt{g L_V / V_N}$$
$$\zeta_P = \frac{D_V}{2\sqrt{g L_V / V_N}}$$

Effect of Airspeed on Phugoid Natural Frequency and Period

Neglecting compressibility effects

$$g\frac{L_V}{V_N} \approx \frac{g}{m}\left[C_{L_N}\rho_N S\right]$$
$$= \frac{2g}{mV_N^2}\left[C_{L_N}\frac{1}{2}\rho_N V_N^2 S\right] = \frac{2g}{mV_N^2}\left[mg\right] = \frac{2g^2}{V_N^2}$$

$$\omega_{n_P} \approx \sqrt{2}\, g \big/ V_N \approx \frac{13.87}{V_N\,(\text{m/s})},\ \text{rad/s}$$

$$T_P = 2\pi/\omega_n, \text{ Period}$$
$$\approx \sqrt{2}\pi V_N / g \approx 0.45 V_N\,(\text{m/s}), \text{sec}$$

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Effect of L/D on Phugoid Damping Ratio

Neglecting compressibility effects and thrust sensitivity to velocity, T_V

$$D_V \approx \frac{1}{m}\left[C_{D_N}\rho_N V_N S\right]$$

$$\zeta = \frac{D_V}{2\sqrt{gL_V / V_N}}$$

$$\approx \frac{C_{D_N}\rho_N V_N S/m}{2\sqrt{2}\, g/V_N} = \frac{C_{D_N}\rho_N V_N^2 S/2}{\sqrt{2}mg} = \frac{1}{\sqrt{2}}\left(\frac{C_{D_N}}{C_{L_N}}\right)$$

$$\zeta \approx \frac{1}{\sqrt{2}(L/D)_N}$$

Velocity m/s	Natural Frequency rad/s	Period sec	L/D	Damping Ratio
50	0.28	23	5	0.14
100	0.14	45	10	0.07
200	0.07	90	20	0.035
400	0.035	180	40	0.018

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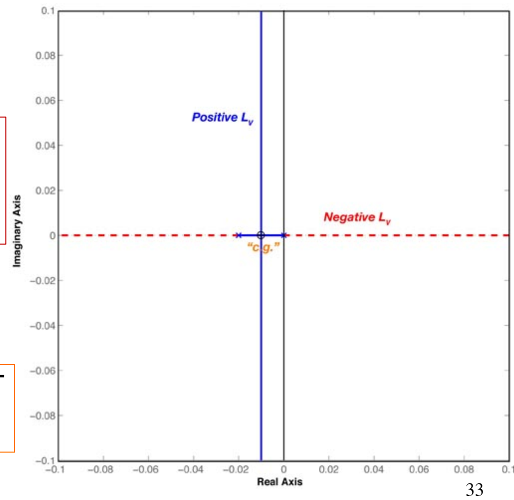
Effect of L_V/V_N Variation on Phugoid Roots

$k = gL_V/V_N$

$$\Delta(s) = \left(s^2 + D_V s\right) + k = \left[s\left(s + D_V\right)\right] + [k]$$

Change in damped natural frequency

$$\omega_{n_{damped}} \triangleq \omega_n \sqrt{1 - \zeta^2}$$



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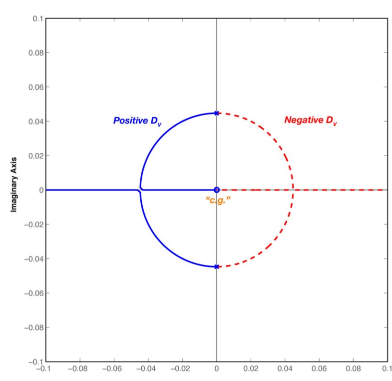
Effect of D_V Variation on Phugoid Roots

$k = D_V$

$$\Delta(s) = \left(s^2 + gL_V/V_N\right) + ks = \left[\left(s + j\sqrt{gL_V/V_N}\right)\left(s - j\sqrt{gL_V/V_N}\right)\right] + [ks]$$

Change in damping ratio

$$\zeta$$



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Approximate Short-Period Model

Approximate Short-Period Equation ($L_q = 0$)

$$\Delta\dot{\mathbf{x}}_{SP} = \begin{bmatrix} \Delta\dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} \approx \begin{bmatrix} M_q & M_\alpha \\ 1 & -L_\alpha / V_N \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta\alpha \end{bmatrix} + \begin{bmatrix} M_{\delta E} \\ -L_{\delta E} / V_N \end{bmatrix} \Delta\delta E$$

Characteristic polynomial

$$\Delta(s) = s^2 + \left(\frac{L_\alpha}{V_N} - M_q\right)s - \left(M_\alpha + M_q\frac{L_\alpha}{V_N}\right)$$
$$= s^2 + 2\zeta\omega_n s + \omega_n^2$$

$$\omega_n = \sqrt{-\left(M_\alpha + M_q\frac{L_\alpha}{V_N}\right)}; \quad \zeta = \frac{\left(\frac{L_\alpha}{V_N} - M_q\right)}{2\sqrt{-\left(M_\alpha + M_q\frac{L_\alpha}{V_N}\right)}}$$

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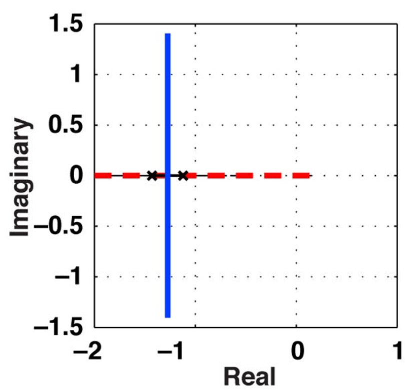
Effect of M_α on Approximate Short-Period Roots

$k = M_\alpha$

$$\Delta(s) = s^2 + \left(\frac{L_\alpha}{V_N} - M_q\right)s - \left(M_q\frac{L_\alpha}{V_N}\right) - k = 0$$
$$= \left(s + \frac{L_\alpha}{V_N}\right)\left(s - M_q\right) - k = 0$$

Change in damped natural frequency



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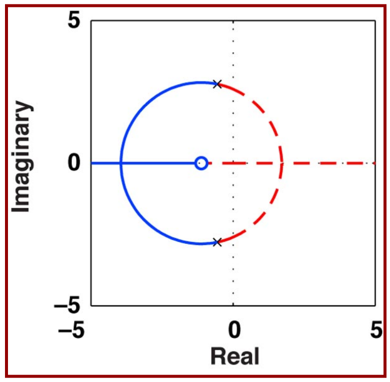
Effect of M_q on Approximate Short-Period Roots

$k = M_q$

Change primarily in damping ratio

$$\Delta(s) = s^2 + \frac{L_\alpha}{V_N}s - M_\alpha - M_q\left(s + \frac{L_\alpha}{V_N}\right)$$



$$\Delta(s) = \left\{s - \left[\frac{L_\alpha}{2V_N} + \sqrt{\left(\frac{L_\alpha}{2V_N}\right)^2 + M_\alpha}\right]\right\}\left\{s - \left[\frac{L_\alpha}{2V_N} - \sqrt{\left(\frac{L_\alpha}{2V_N}\right)^2 + M_\alpha}\right]\right\} - k\left(s + \frac{L_\alpha}{V_N}\right) = 0$$

(–) (–)

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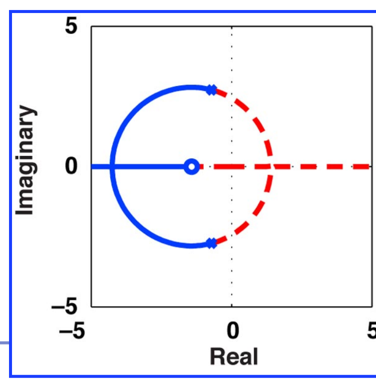
Effect of L_α/V_N on Approximate Short-Period Roots

$k = L_\alpha/V_N$

- Change primarily in damping ratio



$$\Delta(s) = s^2 - M_q s - M_\alpha + \frac{L_\alpha}{V_N}\left(s - M_q\right)$$

$$= \left\{s + \left[\frac{M_q}{2} - \sqrt{\left(\frac{M_q}{2}\right)^2 + M_\alpha}\right]\right\}\left\{s + \left[\frac{M_q}{2} - \sqrt{\left(\frac{M_q}{2}\right)^2 + M_\alpha}\right]\right\} + k\left(s - M_q\right) = 0$$

(–) (–)

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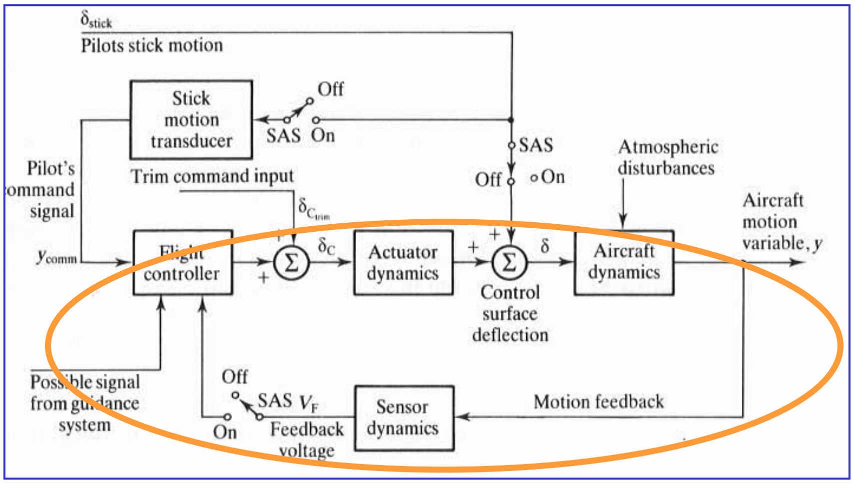
Flight Control Systems



SAS = Stability Augmentation System

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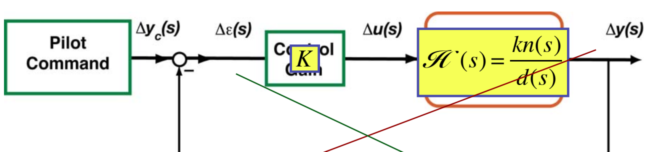
Effect of Scalar Feedback Control on Roots of the System



"Block diagram algebra"

$$\Delta y(s) = \mathcal{H}(s)\Delta u(s) = \frac{kn(s)}{d(s)}\Delta u(s) = \frac{kn(s)}{d(s)}K\Delta\varepsilon(s)$$

$$= K\mathcal{H}(s)\left[\Delta y_c(s) - \Delta y(s)\right]$$

$$\Delta y(s) = K\mathcal{H}(s)\Delta y_c(s) - K\mathcal{H}(s)\Delta y(s)$$

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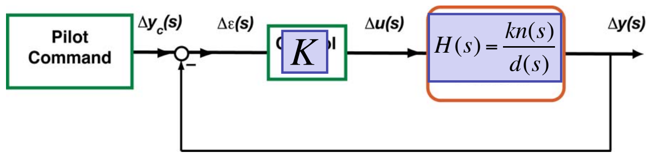
Scalar Closed-Loop Transfer Function



$$\left[1+K\mathscr{H}(s)\right]\Delta y(s)=K\mathscr{H}(s)\Delta y_c(s)$$

$$\frac{\Delta y(s)}{\Delta y_c(s)}=\frac{K\mathscr{H}(s)}{\left[1+K\mathscr{H}(s)\right]}$$

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Roots of the Closed-Loop Control System

$$\frac{\Delta y(s)}{\Delta y_c(s)}=\frac{K\frac{kn(s)}{d(s)}}{\left[1+K\frac{kn(s)}{d(s)}\right]}=\frac{Kkn(s)}{\left[d(s)+Kkn(s)\right]}=\frac{Kkn(s)}{\Delta_{\substack{closed\\loop}}(s)}$$

Closed-loop roots are solutions to

$$\Delta_{\substack{closed-\\loop}}(s)=d(s)+Kkn(s)=0$$

or

$$K\frac{kn(s)}{d(s)}=-1$$

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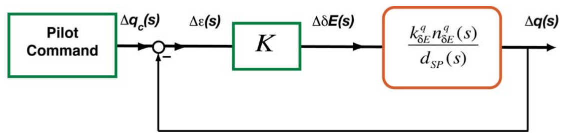
Pitch Rate Feedback to Elevator



$$K\mathscr{H}(s) = K\frac{\Delta q(s)}{\Delta\delta E(s)} = K\frac{k^q_{\delta E}\left(s - z^q_{\delta E}\right)}{s^2 + 2\zeta_{SP}\omega_{n_{SP}}s + \omega_{n_{SP}}^2} = -1$$

- # of roots = 2
- # of zeros = 1
- Destinations of roots (for $k = \pm\infty$):
  - *1 root* goes to zero of *n(s)*
  - *1 root* goes to infinite radius
- Angles of asymptotes, θ, for the roots going to ∞
  - $K \to +\infty$: –180 deg
  - $K \to -\infty$: 0 deg

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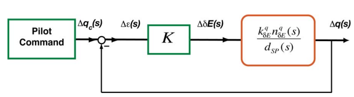
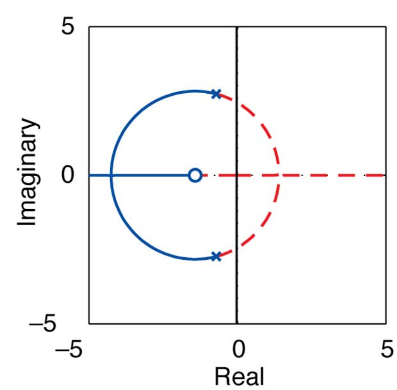
Pitch Rate Feedback to Elevator



- "Center of gravity" on real axis
- Locus on real axis
  - $K > 0$: Segment to the left of the zero
  - $K < 0$: Segment to the right of the zero

Feedback effect is analogous to changing M_q



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Next Time:
Advanced Longitudinal Dynamics

Flight Dynamics
204-206, 503-525

Learning Objectives

Angle-of-attack-rate aero effects
Fourth-order dynamics
Steady-state response to control
Transfer functions
Frequency response
Root locus analysis of parameter variations
Nichols chart
Pilot-aircraft interactions

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Supplemental Material

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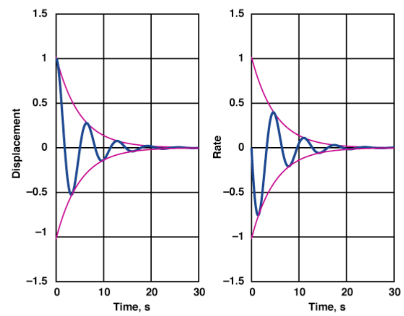
Corresponding 2nd-Order Initial Condition Response

Same envelopes for displacement and rate



$$x_1(t) = Ae^{-\zeta\omega_n t}\sin\left[\omega_n\sqrt{1-\zeta^2}\,t+\varphi\right]$$

$$x_2(t) = Ae^{-\zeta\omega_n t}\left[\omega_n\sqrt{1-\zeta^2}\right]\cos\left[\omega_n\sqrt{1-\zeta^2}\,t+\varphi\right]$$

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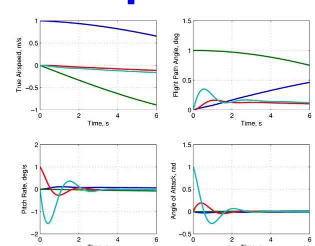
Multi-Modal LTI Responses Superpose Individual Modal Responses

- **With distinct roots, (*n* = 4) for example, *partial fraction expansion* for each state element is**



$$\Delta x_i(s) = \frac{d_{1_i}}{(s-\lambda_1)} + \frac{d_{2_i}}{(s-\lambda_2)} + \frac{d_{3_i}}{(s-\lambda_3)} + \frac{d_{4_i}}{(s-\lambda_4)}, \quad i = 1,4$$

Corresponding 4th-order time response is

$$\Delta x_i(t) = d_{1_i}e^{\lambda_1 t} + d_{2_i}e^{\lambda_2 t} + d_{3_i}e^{\lambda_3 t} + d_{4_i}e^{\lambda_4 t}, \quad i = 1,4$$

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Magnitudes of Roots Going to Infinity

R(+) R(–)

$$s^{(n-q)} = Re^{-j180°} \rightarrow \infty \quad or \quad Re^{-j360°} \rightarrow -\infty$$

$$s = R^{1/(n-q)}\, e^{-j180°/(n-q)} \rightarrow \infty$$

$$or \quad R^{1/(n-q)}\, e^{-j360°/(n-q)} \rightarrow -\infty$$

<u>Magnitudes</u> of roots are the same for given *k*
<u>Angles</u> from the origin are different

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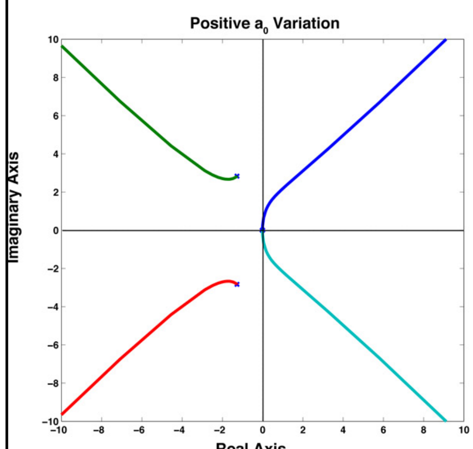
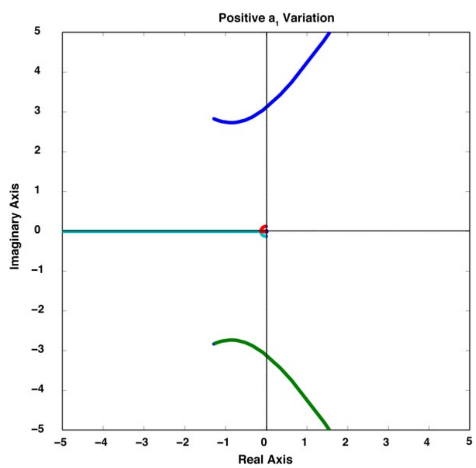
Asymptotes of Roots (for *k* -> ±∞)

4 roots to infinite radius



Asymptotes = ±45, ±135˚

3 roots to infinite radius



Asymptotes = ±60, –180˚

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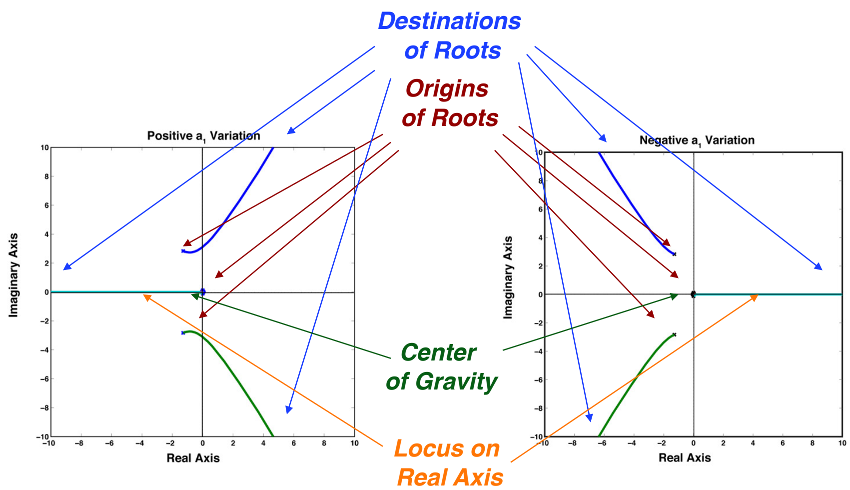
Summary of Root Locus Concepts



Effects of Airspeed, Altitude, Mass, and Moment of Inertia on Fighter Aircraft Short Period

Airspeed variation at constant altitude

Airspeed	Dynamic Pressure	Angle of Attack	Natural Frequency	Period	Damping Ratio
m/s	P	deg	rad/s	sec	-
91	2540	14.6	1.34	4.7	0.3
152	7040	5.8	2.3	2.74	0.31
213	13790	3.2	3.21	1.96	0.3
274	22790	2.2	3.84	1.64	0.3

Mass variation at constant altitude

Mass Variation	Natural Frequency	Period	Damping Ratio
%	rad/s	sec	-
-50	2.4	2.62	0.44
0	2.3	2.74	0.31
50	2.26	2.78	0.26

Altitude variation with constant dynamic pressure

Airspeed	Altitude	Natural Frequency	Period	Damping Ratio
m/s	m	rad/s	sec	-
122	2235	2.36	2.67	0.39
152	6095	2.3	2.74	0.31
213	11915	2.24	2.8	0.23
274	16260	2.18	2.88	0.18

Moment of inertia variation at constant altitude

Moment of Inertia Variation	Natural Frequency	Period	Damping Ratio
%	rad/s	sec	-
-50	3.25	1.94	0.33
0	2.3	2.74	0.31
50	1.87	3.35	0.31