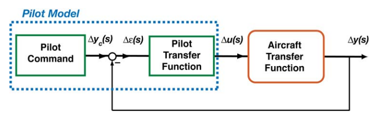
Advanced Problems of Longitudinal Dynamics

Robert Stengel, Aircraft Flight Dynamics
MAE 331, 2018

Learning Objectives

- Fourth-order dynamics
  - Steady-state response to control
  - Transfer functions
  - Frequency response
  - Root locus analysis of parameter variations
- Angle-of-attack-rate aero effects
- Nichols chart
- Pilot-aircraft interactions



Flight Dynamics
204-206, 503-525

http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

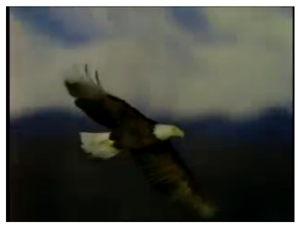
Primary and Coupling Blocks of the Fourth-Order Longitudinal Model

$$\mathbf{F}_{Lon} = \left[\begin{array}{cc|cc} -D_V & -g & 0 & -D_\alpha \\ L_V/V_N & 0 & 0 & L_\alpha/V_N \\ \hline M_V & 0 & M_q & M_\alpha \\ -L_V/V_N & 0 & 1 & -L_\alpha/V_N \end{array}\right] = \begin{bmatrix} \mathbf{F}_{Ph} & \mathbf{F}_{SP}^{Ph} \\ \mathbf{F}_{Ph}^{SP} & \mathbf{F}_{SP} \end{bmatrix}$$

- Some stability derivatives appear only in primary blocks (D_V, M_q, M_α)
  - Effects are well-described by 2nd-order models
- Some stability derivatives appear only in coupling blocks (M_V, D_α)
  - Effects are ignored by 2nd-order models
- Some stability derivatives appear in both (L_V, L_α)
  - May require 4th-order modeling

Why would stability derivatives change?

- Flight condition (airspeed, altitude, ...)
- Fuel and payload variation
  - Center of mass
  - Weight
  - Moments and products of inertia
- Variable geometry ("morphing")
- Aeroelasticity



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How do the 4th-order roots vary when we change pitch-rate damping, M_q?

Identify M_q terms in the characteristic polynomial

$$
\begin{aligned}
\Delta_{Lon}(s) = &\left\{ s^4 + \left(D_V + \frac{L_\alpha}{V_N}\right)s^3 - \left[\left(D_\alpha - g\right)\frac{L_V}{V_N} - D_V\left(\frac{L_\alpha}{V_N}\right) + M_\alpha\right]s^2 \right.\\
&\left. + \left(D_\alpha M_V - D_V M_\alpha\right)s + g\left(M_V \frac{L_\alpha}{V_N} - M_\alpha \frac{L_V}{V_N}\right)\right\}\\
&+\left\{-M_q s^3 - M_q\left(D_V + \frac{L_\alpha}{V_N}\right)s^2 + M_q\left[\left(D_\alpha - g\right)\frac{L_V}{V_N} - D_V\frac{L_\alpha}{V_N}\right]s\right\}\\
&\triangleq d(s) + kn(s)
\end{aligned}
$$

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How do the 4th-order roots vary when we change pitch-rate damping, M_q?

- **Factor terms that are multiplied by M_q to find 3 zeros**
  - **2 zeros near origin similar to approximate phugoid roots, effectively canceling M_q effect on them**

$$-M_q \frac{s\left\{s^2+\left(D_V+\frac{L_\alpha}{V_N}\right)s-\left[(D_\alpha-g)\frac{L_V}{V_N}-D_V\frac{L_\alpha}{V_N}\right]\right\}}{\left[\begin{array}{c} s^2\left\{s^2+\left(D_V+\frac{L_\alpha}{V_N}\right)s-\left[(D_\alpha-g)\frac{L_V}{V_N}-D_V\left(\frac{L_\alpha}{V_N}\right)+M_\alpha\right]\right\} \\ +\left\{D_\alpha M_V-D_V M_\alpha\right\}s+g\left(M_V\frac{L_\alpha}{V_N}-M_\alpha\frac{L_V}{V_N}\right)\end{array}\right]}=-1$$

$$-M_q \frac{s(s-z_1)(s-z_2)}{\left(s^2+2\zeta_P\omega_{n_P}s+\omega_{n_P}^2\right)\left(s^2+2\zeta_{SP}\omega_{n_{SP}}s+\omega_{n_{SP}}^2\right)}=-1$$

Approximate canceling of numerator and denominator terms

$$s(s-z_1)=\left(s^2-z_1 s+0\right)\cong\left(s^2+2\zeta_P\omega_{n_P}s+\omega_{n_P}^2\right)$$

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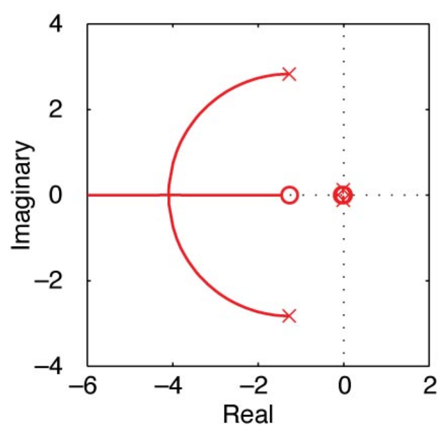
M_q variation has virtually no effect on phugoid roots

$$-M_q \frac{\left(s^2-z_1 s\right)(s-z_2)}{\left(s^2+2\zeta_P\omega_{n_P}s+\omega_{n_P}^2\right)\left(s^2+2\zeta_{SP}\omega_{n_{SP}}s+\omega_{n_{SP}}^2\right)} \simeq \frac{-M_q(s-z_2)}{\left(s^2+2\zeta_{SP}\omega_{n_{SP}}s+\omega_{n_{SP}}^2\right)}=-1$$

Effect on short-period roots is predicted by 2nd-order model



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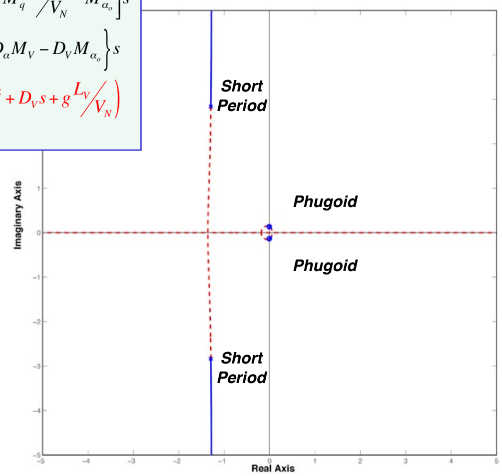
M_α Effect on 4th-Order Roots

Group all terms multiplied by M_α to form numerator for M_α

$$
\begin{aligned}
\Delta_{Lon}(s) = {} & s^4 + \left(D_V + L_\alpha / V_N - M_q\right)s^3 \\
& + \left[(g - D_\alpha) L_V / V_N + D_V\left(L_\alpha / V_N - M_q\right) - M_q L_\alpha / V_N - M_{\alpha_o}\right]s^2 \\
& + \left\{M_q\left[(D_\alpha - g) L_V / V_N - D_V L_\alpha / V_N\right] + D_\alpha M_V - D_V M_{\alpha_o}\right\}s \\
& + g\left(M_V L_\alpha / V_N - M_{\alpha_o} L_V / V_N\right) - \Delta M_\alpha\left(s^2 + D_V s + g L_V / V_N\right) \\
\equiv {} & d(s) + kn(s)
\end{aligned}
$$



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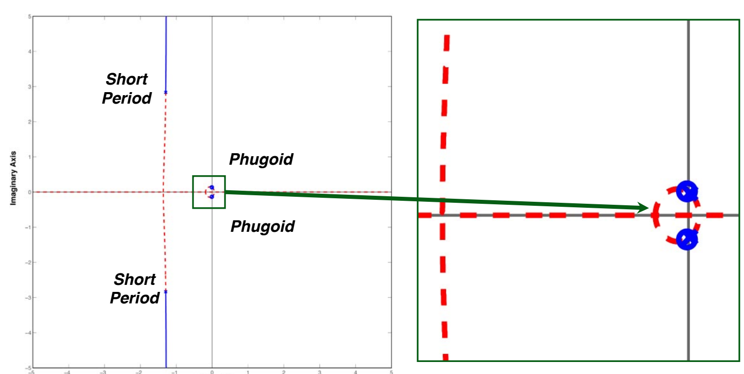
M_α Effect on 4th-Order Roots

- **<u>Primary effect</u>: Same as the approximate short-period model**
- **Numerator zeros**
  - **Same as the approximate phugoid mode characteristic polynomial**
  - **Effect of M_α variation on phugoid mode is small**



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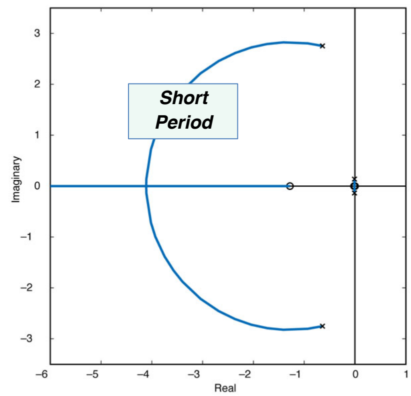
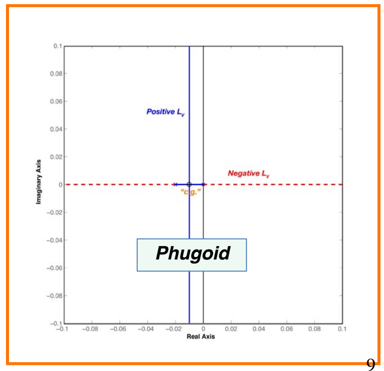
L_α / V_N and L_V / V_N Effects on 4th-Order Roots

- L_α / V_N: Increased damping of the short-period
- Small effect on the phugoid mode



- L_V / V_N: Damped natural frequency of the phugoid
- Negligible effect on the short-period



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Pitching Moment Due to Thrust, M_V

- Thrust line above or below center of mass induces a pitching moment
- Aerodynamic and thrust pitching moments sensitive to velocity perturbation
- Couples phugoid and short-period modes

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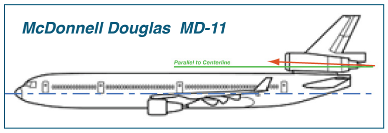
Pitching Moment Due to Thrust, M_V



- Negative $\partial M/\partial V$ (Pitch-down effect) tends to increase velocity
- Positive $\partial M/\partial V$ (Pitch-up effect) tends to decrease velocity
- With propeller thrust line above the *c.m.*, increased velocity decreases thrust, producing a pitch-up moment
- Tilting the thrust line can have benefits
  - Up: Lake Amphibian, MD-11
  - Down: F6F, F8F, AD-1

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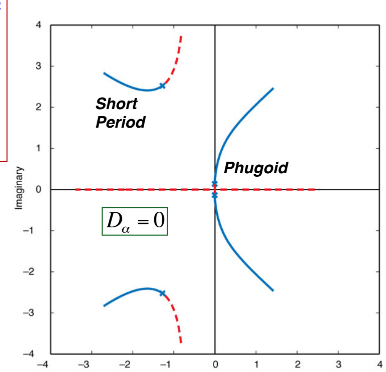
Pitching Sensitivity to Velocity Perturbations (M_V)

$$
\begin{aligned}
\Delta_{Lon}(s) = 0 = {} & s^4 + \left(D_V + \frac{L_\alpha}{V_N} - M_q\right)s^3 \\
& + \left[(g - D_\alpha)\frac{L_V}{V_N} + D_V\left(\frac{L_\alpha}{V_N} - M_q\right) - M_q\frac{L_\alpha}{V_N} - M_\alpha\right]s^2 \\
& + \left\{M_q\left[(D_\alpha - g)\frac{L_V}{V_N} - D_V\frac{L_\alpha}{V_N}\right] + D_\alpha M_V - D_V M_\alpha\right\}s \\
& + gM_\alpha\frac{L_V}{V_N} + M_V\left(D_\alpha s + g\frac{L_\alpha}{V_N}\right)
\end{aligned}
$$



- Large positive value produces oscillatory phugoid instability
- Large negative value produces real phugoid divergence
- How do you solve the problem?
- Increase phugoid damping (e.g., D_V)

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4^{th}-Order Frequency Response and Feedback Control Effects

Longitudinal Model Transfer Function Matrix (H_x = I, H_u = 0)

$$\begin{bmatrix} \Delta V(s) \\ \Delta\gamma(s) \\ \Delta q(s) \\ \Delta\alpha(s) \end{bmatrix} = \mathbf{H_x}\left[s\mathbf{I}-\mathbf{F}\right]^{-1}\mathbf{G}\begin{bmatrix} \Delta\delta E(s) \\ \Delta\delta T(s) \\ \Delta\delta F(s) \end{bmatrix} = \mathscr{H}_{Lon}(s)\begin{bmatrix} \Delta\delta E(s) \\ \Delta\delta T(s) \\ \Delta\delta F(s) \end{bmatrix}$$

$$\mathscr{H}_{Lon}(s) = \frac{\begin{bmatrix} n_{\delta E}^{V}(s) & n_{\delta T}^{V}(s) & n_{\delta F}^{V}(s) \\ n_{\delta E}^{\gamma}(s) & n_{\delta T}^{\gamma}(s) & n_{\delta F}^{\gamma}(s) \\ n_{\delta E}^{q}(s) & n_{\delta T}^{q}(s) & n_{\delta F}^{q}(s) \\ n_{\delta E}^{\alpha}(s) & n_{\delta T}^{\alpha}(s) & n_{\delta F}^{\alpha}(s) \end{bmatrix}}{\left(s^2+2\zeta_P\omega_{n_P}s+\omega_{n_P}^2\right)\left(s^2+2\zeta_{SP}\omega_{n_{SP}}s+\omega_{n_{SP}}^2\right)}$$

Transfer Functions of Elevator Input to Angle Output*

Elevator-to-Flight Path Angle transfer function

$$\frac{\Delta\gamma(s)}{\Delta\delta E(s)} = \frac{n^{\gamma}_{\delta E}(s)}{\Delta_{Lon}(s)}; \quad n^{\gamma}_{\delta E}(s) = M_{\delta E}\frac{L_\alpha}{V_N}\left(s + \frac{1}{T_{\gamma_1}}\right)$$

Elevator-to-Angle of Attack transfer function

$$\frac{\Delta\alpha(s)}{\Delta\delta E(s)} = \frac{n^{\alpha}_{\delta E}(s)}{\Delta_{Lon}(s)}; \quad n^{\alpha}_{\delta E}(s) = M_{\delta E}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{Approx\ Ph}$$

Elevator-to-Pitch Angle transfer function

$$\frac{\Delta\theta(s)}{\Delta\delta E(s)} = \frac{n^{\theta}_{\delta E}(s)}{\Delta_{Lon}(s)}; \quad n^{\theta}_{\delta E}(s) = M_{\delta E}\left(s + \frac{1}{T_{\theta_1}}\right)\left(s + \frac{1}{T_{\theta_2}}\right)$$

* Flying qualities notation for numerator zero time constants

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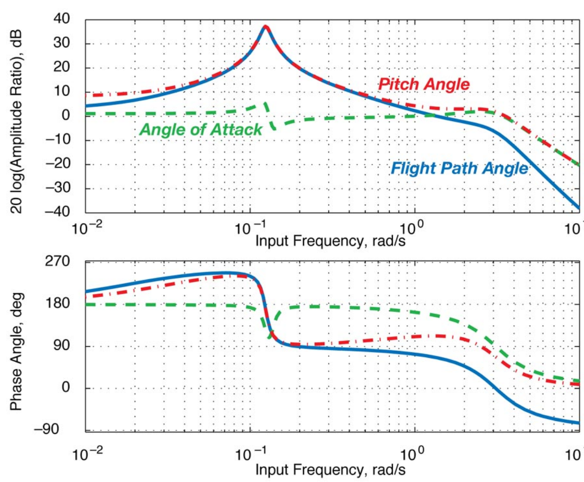
Frequency Response of Angles to Elevator Input

- **Pitch angle frequency response ($\Delta\theta = \Delta\gamma + \Delta\alpha$)**
  - **Similar to flight path angle near phugoid natural frequency**
  - **Similar to angle of attack near short-period natural frequency**



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Apparent Mass and Unsteady Aerodynamics

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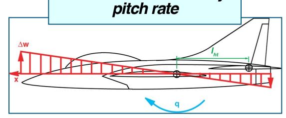
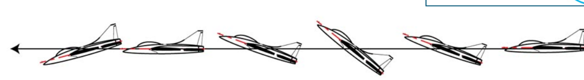
Distinction Between Angle-of-Attack Rate and Pitch Rate

Vertical velocity distribution induced by pitch rate



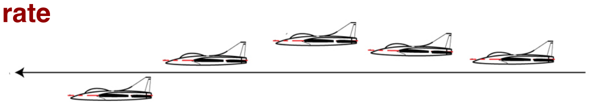
- With no vertical motion of the *c.m.*, pitch rate and angle-of-attack rate are the same

$$\dot{\alpha} = q$$



- With no pitching, vertical heaving (or plunging) motion of the *c.m.* produces angle-of-attack rate but no pitch rate

$$\dot{\alpha} \neq 0; \quad q = 0$$



Could $\dot{\alpha} = 0; \quad q \neq 0$ **?**

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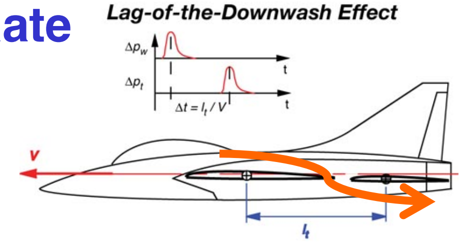
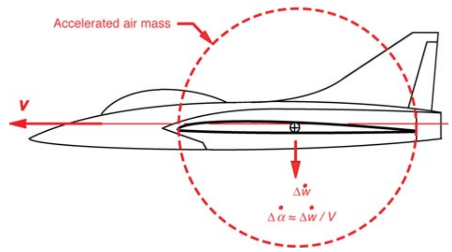
Angle-of-Attack Rate Has Two Effects

- Pressure variations at wing convect downstream, arriving at tail **Δ*t* sec** later
  - Lag of the downwash
  - Delayed tail-lift/pitch-moment effect
- Vertical force opposed by a mass of air (apparent mass) as well as airplane mass
  - Vertical acceleration produces added lift and moment

Lag-of-the-Downwash Effect



Apparent Mass Effect



***Flight Dynamics*, pp. 204-206, 284-285**

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Angle-of-Attack-Rate Effects Principally Affect the Short-Period Mode

Lift and pitching moment proportional to angle-of-attack rate

$$\Delta \dot{q} = M_q \Delta q + M_\alpha \Delta \alpha + M_{\delta E} \Delta \delta E + M_{\dot{\alpha}} \Delta \dot{\alpha}$$

$$\Delta \dot{\alpha} = \left(1 - \frac{L_q}{V_N}\right)\Delta q - \left(\frac{L_\alpha}{V_N}\right)\Delta \alpha - \left(\frac{L_{\delta E}}{V_N}\right)\Delta \delta E - \left(\frac{L_{\dot{\alpha}}}{V_N}\right)\Delta \dot{\alpha}$$

Bring $\dot{\alpha}$ effects to left side

$$\Delta \dot{q} - M_{\dot{\alpha}} \Delta \dot{\alpha} = M_q \Delta q + M_\alpha \Delta \alpha + M_{\delta E} \Delta \delta E$$

$$\Delta \dot{\alpha} + \left(\frac{L_{\dot{\alpha}}}{V_N}\right)\Delta \dot{\alpha} = \left(1 - \frac{L_q}{V_N}\right)\Delta q - \left(\frac{L_\alpha}{V_N}\right)\Delta \alpha - \left(\frac{L_{\delta E}}{V_N}\right)\Delta \delta E$$

Vector-matrix form

$$\begin{bmatrix} 1 & -M_{\dot{\alpha}} \\ 0 & \left[1 + \left(\frac{L_{\dot{\alpha}}}{V_N}\right)\right] \end{bmatrix} \begin{bmatrix} \Delta \dot{q} \\ \Delta \dot{\alpha} \end{bmatrix} = \begin{bmatrix} M_q & M_\alpha \\ \left(1 - \frac{L_q}{V_N}\right) & -\left(\frac{L_\alpha}{V_N}\right) \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta \alpha \end{bmatrix} + \begin{bmatrix} M_{\delta E} \\ -\left(\frac{L_{\delta E}}{V_N}\right) \end{bmatrix} \Delta \delta E$$

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Solve for $\Delta\dot{q}$ and $\Delta\dot{\alpha}$

Multiply both sides by the inverse

$$\begin{bmatrix}\Delta\dot{q}\\ \Delta\dot{\alpha}\end{bmatrix}=\begin{bmatrix}1 & -M_{\dot{\alpha}}\\ 0 & \left[1+\left(L_{\dot{\alpha}}/V_N\right)\right]\end{bmatrix}^{-1}\left\{\begin{bmatrix}M_q & M_\alpha\\ \left(1-L_q/V_N\right) & -\left(L_\alpha/V_N\right)\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta\alpha\end{bmatrix}+\begin{bmatrix}M_{\delta E}\\ -\left(L_{\delta E}/V_N\right)\end{bmatrix}\Delta\delta E\right\}$$

$$\begin{bmatrix}\Delta\dot{q}\\ \Delta\dot{\alpha}\end{bmatrix}\simeq\begin{bmatrix}\left\{M_q+M_{\dot{\alpha}}\right\} & \left\{M_\alpha-M_{\dot{\alpha}}\left(L_\alpha/V_N\right)\right\}\\ 1 & -\left(L_\alpha/V_N\right)\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta\alpha\end{bmatrix}+\begin{bmatrix}M_{\delta E}-M_{\dot{\alpha}}\left(L_{\delta E}/V_N\right)\\ -\left(L_{\delta E}/V_N\right)\end{bmatrix}\Delta\delta E$$

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Control Surface Dynamic Coupling

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Dynamic Model of a Control Surface Mechanism

Approximate control dynamics by a 2nd-order LTI system

$$\delta \ddot{E} = \left(H_{aero} + H_{control}\right) / \left(Mechanical\ Inertia\right)$$

Elevator Example

$$\ddot{\delta E} = H_{elevator} / I_{elevator} = C_{H_{elevator}} \frac{1}{2} \rho V^2 S\bar{c} \Big/ I_{elevator}$$

$$= \left[C_{H_{\dot{\delta E}}} \dot{\delta E} + C_{H_{\delta E}} \delta E + C_{H_{\alpha}} \alpha + C_{H_{command}} + \ldots \right] \rho V^2 S\bar{c} / 2 I_{elevator}$$

$$\triangleq H_{\dot{\delta E}} \dot{\delta E} + H_{\delta E} \delta E + H_{\alpha} \alpha + H_{command} + \ldots$$

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Coupling of System Model and Control Mechanism Dynamics

- **2nd-order model of control-deflection dynamics**
  - Forcing by aerodynamic effects
    - Control surface deflection
    - Aircraft angle of attack and angular rates

$$\Delta \dot{\mathbf{x}}_{\delta E} = \mathbf{F}_{\delta E} \Delta \mathbf{x}_{\delta E} + \mathbf{G}_{\delta E} \Delta \mathbf{u}_{\delta E} + \mathbf{F}_{SP}^{\delta E} \Delta \mathbf{x}_{SP}$$

$$\begin{bmatrix} \Delta \dot{\delta} E \\ \Delta \ddot{\delta} E \end{bmatrix} \approx \begin{bmatrix} 0 & 1 \\ H_{\delta E} & H_{\dot{\delta} E} \end{bmatrix} \begin{bmatrix} \Delta \delta E \\ \Delta \dot{\delta} E \end{bmatrix} + \begin{bmatrix} 0 \\ -H_{\delta E} \end{bmatrix} \Delta \delta E_{command} + \begin{bmatrix} 0 & 0 \\ H_q & H_\alpha \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta \alpha \end{bmatrix}$$

- **Short period approximation**
  - Coupling with mechanism dynamics

$$\Delta \dot{\mathbf{x}}_{SP} = \mathbf{F}_{SP} \Delta \mathbf{x}_{SP} + \mathbf{G}_{SP} \Delta \mathbf{u}_{SP} \triangleq \mathbf{F}_{SP} \Delta \mathbf{x}_{SP} + \mathbf{F}_{\delta E}^{SP} \Delta \mathbf{x}_{\delta E}$$

$$\begin{bmatrix} \Delta \dot{q} \\ \Delta \dot{\alpha} \end{bmatrix} \approx \begin{bmatrix} M_q & M_\alpha \\ 1 & -L_\alpha / V_N \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta \alpha \end{bmatrix} + \begin{bmatrix} M_{\delta E} & 0 \\ -L_{\delta E} / V_N & 0 \end{bmatrix} \begin{bmatrix} \Delta \delta E \\ \Delta \dot{\delta} E \end{bmatrix}$$

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Short Period Model Augmented by Control Mechanism Dynamics

State Vector

$$\Delta \mathbf{x}_{SP'} \triangleq \begin{bmatrix} \Delta q \\ \Delta\alpha \\ \Delta\delta E \\ \Delta\dot{\delta}E \end{bmatrix}$$

Augmented dynamic equation

$$\Delta\dot{\mathbf{x}}_{SP/\delta E} = \mathbf{F}_{SP/\delta E}\Delta\mathbf{x}_{SP/\delta E} + \mathbf{G}_{SP/\delta E}\Delta\delta E_{command}$$

Augmented stability and control matrices

$$\mathbf{F}_{SP/\delta E} = \begin{bmatrix} \mathbf{F}_{SP} & \mathbf{F}_{\delta E}^{SP} \\ \mathbf{F}_{SP}^{\delta E} & \mathbf{F}_{\delta E} \end{bmatrix} = \left[\begin{array}{cc|cc} M_q & M_\alpha & M_{\delta E} & 0 \\ 1 & -L_\alpha/V_N & -L_{\delta E}/V_N & 0 \\ \hline 0 & 0 & 0 & 1 \\ H_q & H_\alpha & H_{\delta E} & H_{\dot{\delta}E} \end{array}\right]$$

$$\mathbf{G}_{SP/\delta E} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ H_{\delta E} \end{bmatrix}$$

Roots of the Augmented Short Period Model

Characteristic equation for short-period/elevator dynamics

$$\Delta_{SP/\delta E}(s) = \left|s\mathbf{I}_n - \mathbf{F}_{SP/\delta E}\right| = \left|\begin{array}{cc|cc} (s - M_q) & -M_\alpha & -M_{\delta E} & 0 \\ -1 & \left(s + L_\alpha/V_N\right) & L_{\delta E}/V_N & 0 \\ \hline 0 & 0 & s & -1 \\ -H_q & -H_\alpha & -H_{\delta E} & (s - H_{\dot{\delta}E}) \end{array}\right| = 0$$

$$\Delta_{SP/\delta E}(s) = \left(s^2 + 2\zeta_{SP}\omega_{n_{SP}}s + \omega_{n_{SP}}^2\right)\left(s^2 + 2\zeta_{\delta E}\omega_{n_{\delta E}}s + \omega_{n_{\delta E}}^2\right)$$

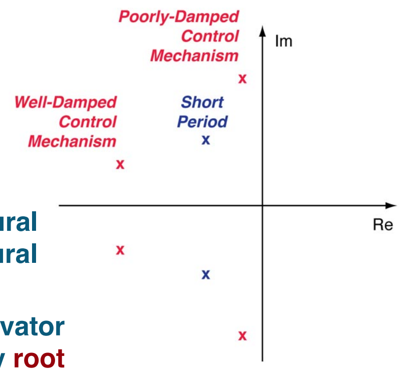
Short Period — Control Mechanism

Roots of the Augmented Short Period Model

- Coupling of the modes depends on design parameters

$$M_{\delta E}, \frac{L_{\delta E}}{V_N}, H_q, \text{ and } H_\alpha$$

- Desirable for mechanical natural frequency > short-period natural frequency
- Coupling dynamics (e.g., "elevator floating") can be evaluated by **root locus analysis**



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Gain and Phase Margins: The Nichols Chart

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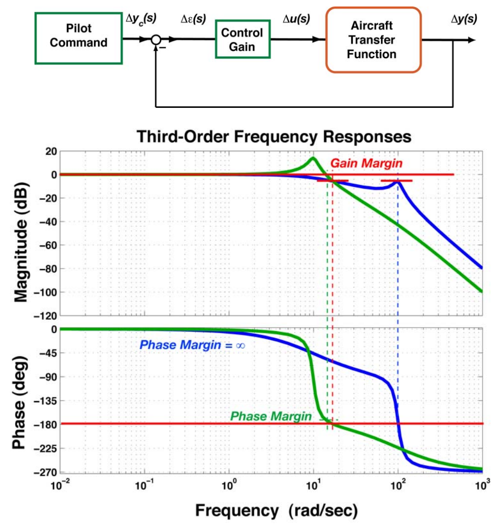
Gain and Phase Margins

- **Gain Margin**
  - Measured at the input frequency, ω, for which $\phi(j\omega) = -180°$
  - Difference between 0 dB and transfer function magnitude, $20 \log_{10} AR(j\omega)$
- **Phase Margin**
  - Measured at the input frequency, ω, for which $20 \log_{10} AR(j\omega) = 0$ dB
  - Difference between the phase angle $\phi(j\omega)$, and $-180°$



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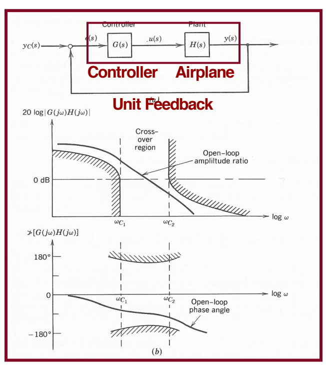
Desirable Open-Loop Frequency Response Characteristics: Crossover Region

$$\mathcal{H}_{open-loop}(s) = \mathcal{H}_{control}(s)\mathcal{H}_{airplane}(s)$$

- **High gain (amplitude) at low frequency**
  - Desired response is slowly varying
- **Low gain at high frequency**
  - Random errors vary rapidly
- **Crossover frequency range is problem-specific**



$$\text{For } \omega_{C_1} < \omega < \omega_{C_1},$$
$$\text{Slope of Open-Loop Amplitude Ratio} \approx -20 \text{ dB / dec}$$
$$\text{Open-Loop Phase Angle} \approx -90°$$

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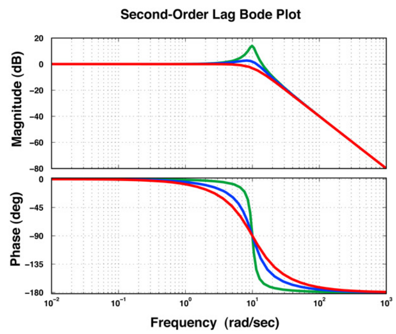
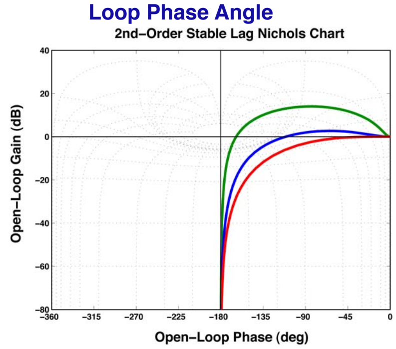
Nichols Chart: Gain vs. Phase Angle

- **Bode Plot**
  - Two plots
  - Open-Loop Gain (dB) vs. $\log_{10}\omega$
  - Open-Loop Phase Angle vs. $\log_{10}\omega$



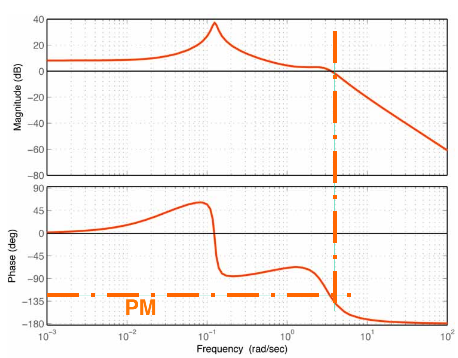
- **Nichols Chart**
  - Single crossplot; input frequency not shown
  - Open-Loop Gain (dB) vs. Open-Loop Phase Angle



Axis intercepts on the Nichols Chart identify GM and PM

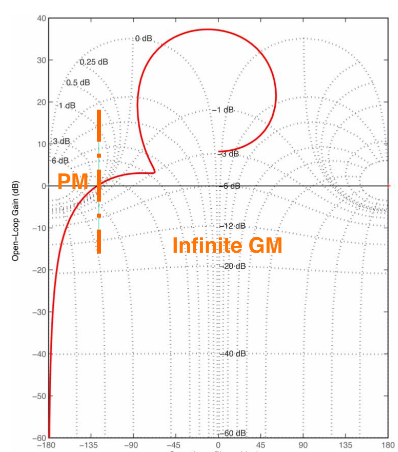
Aircraft Transfer Function for Pitch-Tracking Task

Elevator-to-Pitch-Angle Bode Plot



- **Gain Margin:** Amplitude ratio below 0 dB when phase angle = 180°
- **Phase Margin:** Phase angle above –180° when amplitude ratio = 0 dB

Elevator-to-Pitch-Angle Nichols Chart



Pilot-Vehicle Interactions

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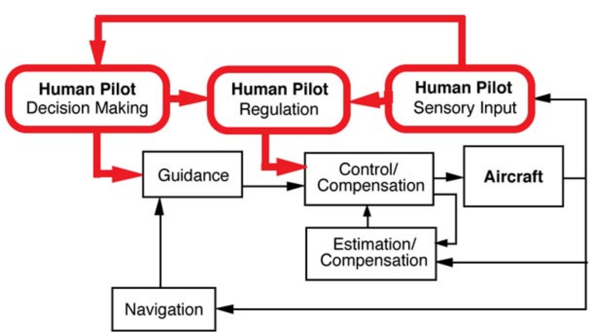
Pilot Inputs to Control



* p. 421-425, *Flight Dynamics*

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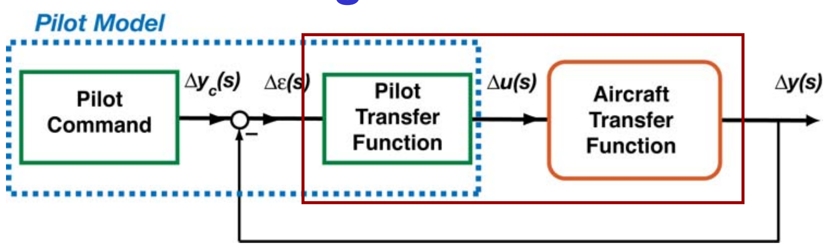
Effect of Pilot Dynamics on Pitch-Angle Control Task

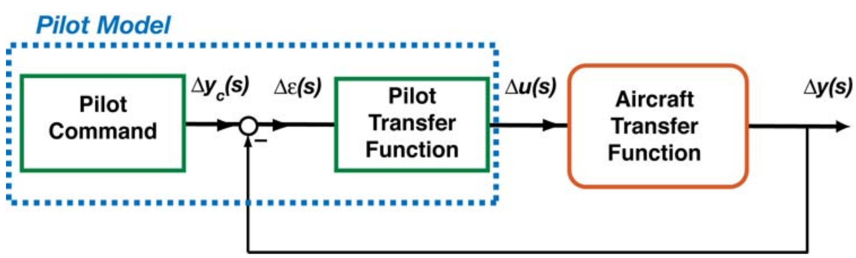


Open-Loop 1st-Order-Pilot/Aircraft Transfer Function

$$\mathcal{H}_{open-loop}(s) = \mathcal{H}_{pilot}(s)\mathcal{H}_{airplane}(s)$$

$$= K_P\left[\frac{1/T_P}{(s+1/T_P)}\right]\left[\frac{M_{\delta E}\left(s+\frac{1}{T_{\theta_1}}\right)\left(s+\frac{1}{T_{\theta_2}}\right)}{\left(s^2+2\zeta\omega_n s+\omega_n^2\right)_{Ph}\left(s^2+2\zeta\omega_n s+\omega_n^2\right)_{SP}}\right]$$

Effect of Pilot Dynamics on Pitch-Angle Control Task

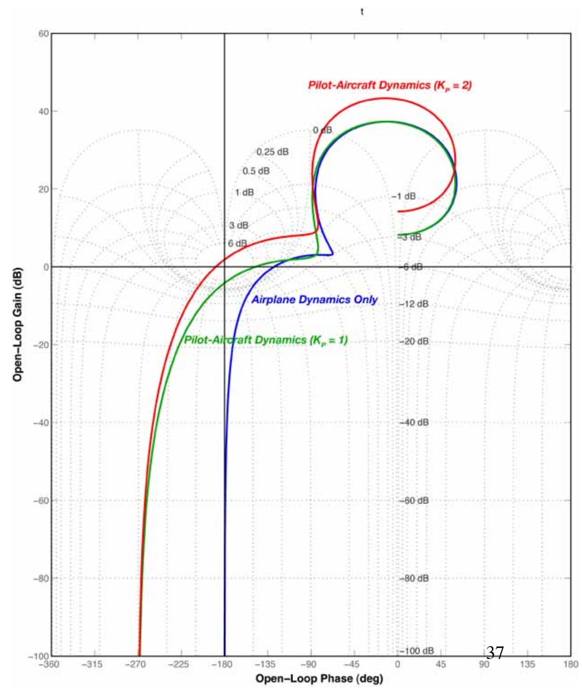


$$\mathcal{H}_{pilot}(s) = \frac{\Delta u(s)}{\Delta\varepsilon(s)} = K_P\frac{1/T_P}{s+1/T_P} = K_P\frac{1/0.25}{s+1/0.25}$$

- **Pilot introduces neuromuscular lag while closing the control loop**
- **Example**
  - **Model the lag by a 1st-order time constant, T_P, of 0.25 sec**
  - **Pilot's gain, K_P, is either 1 or 2**

Effect of Pilot Dynamics on Pitch-Angle Control Task

- **Gain and phase margins become negative for pilot gain, K_P, between 1 and 2**
- **Then, pilot destabilizes the system (PIO)**



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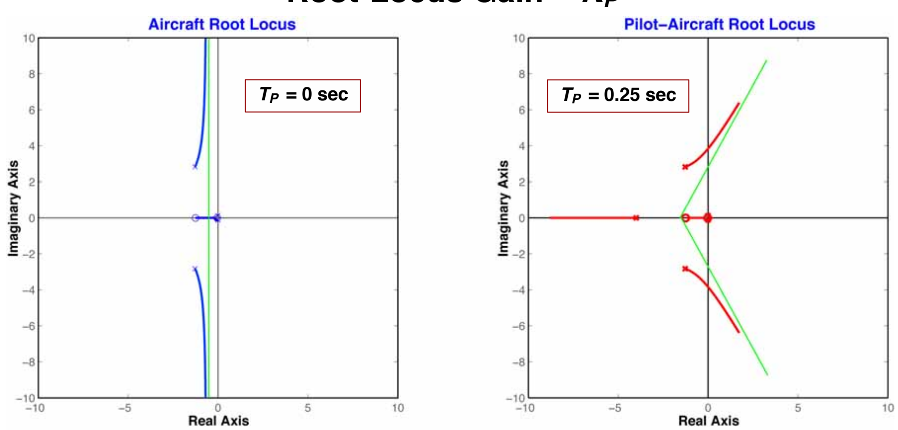
Effect of 1st-Order Pilot Dynamics on Elevator/Pitch-Angle Control Root Locus

Root-Locus Gain = K_P



Pilot's time delay changes asymptotes of the root locus

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Pilot-Induced Oscillations

Uncommanded aircraft is stable but piloting actions couple with aircraft dynamics to produce instability

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Pilot-Induced Oscillations

NASA Digital-Fly-By-Wire F-8
Simulation of Space Shuttle

http://www.dfrc.nasa.gov/Gallery/Movie/F-8DFBW/Medium/EM-0044-01.mov

http://www.dfrc.nasa.gov/Gallery/Movie/F-8DFBW/Medium/EM-0044-02.mov

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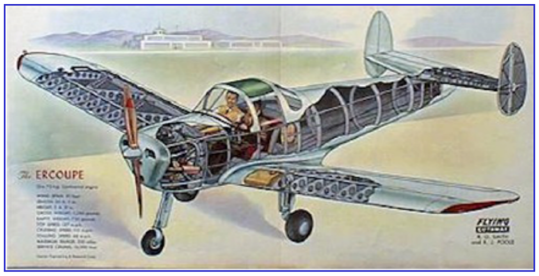
Personal Aircraft Factoids

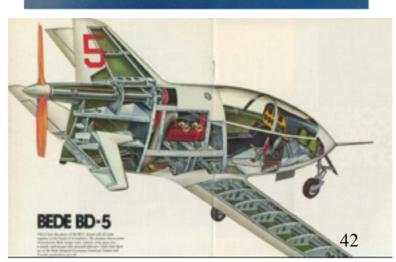
Ercoupe Approach to Safe, Affordable Flying



- Limited center-of-mass travel
- Limited range of airspeed from TO/Landing through cruising flight
- Fixed, tricycle landing gear
- Limited control authority
- Control yoke, no rudder pedals (initially)
- Aileron-rudder interconnect
- Good lateral stability (stable spiral mode)

Less-Safe Airplanes

- *Mignet Flying Flea* (Homebuilt, pivoting main wing, no ailerons, unrecoverable dive, unrecoverable inverted flight)
- V-tail *Beechcraft Bonanza Model 35* (10,000 built, 250 in-flight structural failures)
- *American Yankee AA-1* (BD-1, "hot", stalls and spins)
- *Bede BD-5* (Home-built, unforgiving flying qualities)



Do "Safe" Airplanes Have Fewer Accidents?

- The *Ercoupe's* safety record is about average (1940-1970)
- Many *Piper J3 Cubs* involved in fatal stall/spin accidents (1937-1947)
- *Cirrus SR-20/22*: mixed record, but improving (1997-present)



- Probable cause of Ercoupe fatal accidents: 1994-2004
  - VFR flight in IFR (IMC) conditions
  - Pilot's lack of night-flying experience
  - Pilot had <u>no</u> flight training
  - Inadvertent stall
  - Loss of engine power
  - In-flight breakup (corrosion)

http://home.iwichita.com/rh1/eddy/Safe_Airplane_NOT.htm

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Cessna 172

- <u>*AOPA News*, 1995</u>: "The world's most popular airplane … has a great safety record…. 24,130 *Skyhawks* in the fleet."
  - All Cessna 172 accidents, 1982-1988: 1,600+
  - About 237 reportable C172 accidents/yr
  - Happily, most result in little or no injury.
- <u>*Air Facts*, 2016</u>: Most built GA plane in history: 43,000.
  - 1995-2000: Fatal accident: 0.56 per 100,000 flying hours, best accident rate in private aviation.
  - 2012-13: C172 safety record appears to have improved, maybe substantially.
- <u>*Town Topics*, May 20, 1992</u>:
  - "Two Princeton University students were killed, apparently instantly, early Tuesday morning when their four-seat Cessna 172 plane crashed head-on into a 50-foot tree that borders a clear area beyond the Princeton Airport runway."

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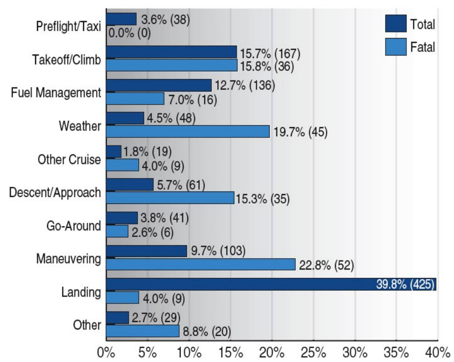
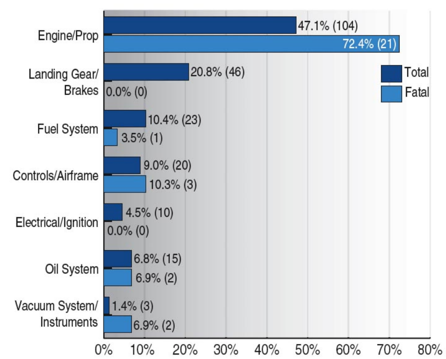
2004 General Aviation Accidents

(AOPA)





https://www.aopa.org/news-and-media/all-news/2016/august/11/nall-report-notes-progress

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How Safe is "Safe"?

(NTSB)

Accidents, Fatalities, and Rates, 1986 through 2005, U.S. General Aviation

	Accidents		Fatalities			Accidents per 100,000 Flight Hours	
Year	All	Fatal	Total	Aboard	Flight Hours	All	Fatal
1986	2,581	474	967	879	27,073,000	9.49	1.73
1987	2,495	446	837	822	26,972,000	9.18	1.63
1988	2,388	460	797	792	27,446,000	8.65	1.66
1989	2,242	432	769	766	27,920,000	7.97	1.52
1990	2,242	444	770	765	28,510,000	7.85	1.55
1991	2,197	439	800	786	27,678,000	7.91	1.57
1992	2,111	451	867	865	24,780,000	8.51	1.82
1993	2,064	401	744	740	22,796,000	9.03	1.74
1994	2,021	404	730	723	22,235,000	9.08	1.81
1995	2,056	413	735	728	24,906,000	8.21	1.63
1996	1,908	361	636	619	24,881,000	7.65	1.45
1997	1,844	350	631	625	25,591,000	7.19	1.36
1998	1,905	365	625	619	25,518,000	7.44	1.41
1999	1,905	340	619	615	29,246,000	6.5	1.16
2000	1,837	345	596	585	27,838,000	6.57	1.21
2001	1,727	325	562	558	25,431,000	6.78	1.27
2002	1,715	345	581	575	25,545,000	6.69	1.33
2003	1,739	352	632	629	25,706,000	6.75	1.36
2004	1,617	314	558	558	24,888,000	6.49	1.26
2005	1,669	321	562	557	24,401,000	6.83	1.31

Accidents, Fatalities, and Rates, 1986 through 2005, for U.S. Air Carriers Operating Under 14 CFR 121

	Accidents		Fatalities			Accidents per 100,000 Flight Hours	
Year	All	Fatal	Total	Aboard	Flight Hours	All	Fatal
1986	21	2	5	4	9,495,158	0.211	0.011
1987	32	4	231	229	10,115,407	0.306	0.03
1988	29	3	285	274	10,521,052	0.266	0.019
1989	23	8	131	130	10,597,922	0.217	0.075
1990	21	6	39	12	11,524,726	0.182	0.052
1991	22	4	62	49	11,139,166	0.198	0.036
1992	16	4	33	31	11,732,026	0.136	0.034
1993	22	1	1	0	11,981,347	0.184	0.008
1994	18	4	239	237	12,292,356	0.138	0.033
1995	30	1	160	160	12,776,679	0.235	0.008
1996	31	3	342	342	12,971,676	0.239	0.023
1997	43	3	3	2	15,061,662	0.285	0.02
1998	41	1	1	0	15,921,447	0.258	0.006
1999	40	2	12	11	16,693,365	0.24	0.012
2000	49	2	89	89	17,478,519	0.28	0.011
2001	41	6	531	525	17,157,858	0.216	0.012
2002	35	0	0	0	16,718,781	0.209	-
2003	51	2	22	21	16,887,756	0.302	0.012
2004	23	1	13	13	18,184,016	0.126	0.005
2005	32	3	22	20	18,728,000	0.171	0.016

… but statistics are variable among sources

http://www.rita.dot.gov/bts/sites/rita.dot.gov.bts/files/publications/national_transportation_statistics/html/table_02_14.html

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Next Time:
Advanced Problems of Lateral-Directional Dynamics

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Supplemental Material

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How do the 4th-order roots vary when we change pitch-rate damping, M_q?

Separate non-M_q and M_q terms

$$d(s) = s^4 + \left(D_V + L_\alpha/V_N\right)s^3 - \left[(D_\alpha - g)L_V/V_N - D_V\left(L_\alpha/V_N\right) + M_\alpha\right]s^2 + \left(D_\alpha M_V - D_V M_\alpha\right)s + g\left(M_V L_\alpha/V_N - M_\alpha L_V/V_N\right)$$

$$kn(s) = -M_q\left\{s^3 + \left[D_V + L_\alpha/V_N\right]s^2 - \left[(D_\alpha - g)L_V/V_N - D_V L_\alpha/V_N\right]s\right\}$$
$$= -M_q s\left\{s^2 + \left[D_V + L_\alpha/V_N\right]s - \left[(D_\alpha - g)L_V/V_N - D_V L_\alpha/V_N\right]\right\}$$

$$\frac{kn(s)}{d(s)} = -1$$

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2nd-Degree Characteristic Polynomial with

$$L_q \text{ and } L_{\dot\alpha} \simeq 0$$

Short-period characteristic polynomial

$$\Delta(s) = \begin{vmatrix} \left[s - \left(M_q + M_{\dot\alpha}\right)\right] & -\left[M_\alpha - M_{\dot\alpha}\left(L_\alpha/V_N\right)\right] \\ -1 & \left[s + \left(L_\alpha/V_N\right)\right] \end{vmatrix} = \left[s - \left(M_q + M_{\dot\alpha}\right)\right]\left[s + \left(L_\alpha/V_N\right)\right] - \left[M_\alpha - M_{\dot\alpha}\left(L_\alpha/V_N\right)\right]$$

$$= s^2 + \left[\left(L_\alpha/V_N\right) - \left(M_q + M_{\dot\alpha}\right)\right]s + \left\{\left[M_\alpha - \left(M_q + M_{\dot\alpha}\right)\left(L_\alpha/V_N\right)\right] + M_{\dot\alpha}\left(L_\alpha/V_N\right)\right\}$$

Damping is increased
Natural frequency is unaffected

$$\Delta(s) = s^2 + \left[\left(L_\alpha/V_N\right) - \left(M_q + M_{\dot\alpha}\right)\right]s + \left\{\left[M_\alpha - M_q\left(L_\alpha/V_N\right)\right]\right\}$$
$$= s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

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Comparison of Bizjet Fourth- and Second-Order Models and Eigenvalues

Fourth-Order Model

F =				G =		Eigenvalue	Damping	Freq. (rad/s)
-0.0185	-9.8067	0	0	0	4.6645	-8.43e-03 + 1.24e-01j	6.78E-02	1.24E-01
0.0019	0	0	1.2709	0	0	-8.43e-03 - 1.24e-01j	6.78E-02	1.24E-01
0	0	-1.2794	-7.9856	-9.069	0	-1.28e+00 + 2.83e+00j	4.11E-01	3.10E+00
-0.0019	0	1	-1.2709	0	0	-1.28e+00 - 2.83e+00j	4.11E-01	3.10E+00

Phugoid Approximation

F =		G =	Eigenvalue	Damping	Freq. (rad/s)
-0.0185	-9.8067	4.6645	-9.25e-03 + 1.36e-01j	6.78E-02	1.37E-01
0.0019	0	0	-9.25e-03 - 1.36e-01j	6.78E-02	1.37E-01

Short-Period Approximation

F =		G =	Eigenvalue	Damping	Freq. (rad/s)
-1.2794	-7.9856	-9.069	-1.28e+00 + 2.83e+00j	4.11E-01	3.10E+00
1	-1.2709	0	-1.28e+00 - 2.83e+00j	4.11E-01	3.10E+00

- Approximations are very close to 4th-order values because natural frequencies are widely separated

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A Little More About Output Matrices

With $\mathbf{H_x} = \mathbf{I}$ and $\mathbf{H_u} = \mathbf{0}$

$$\Delta\mathbf{y} = \Delta\mathbf{x} = \mathbf{H_x}\Delta\mathbf{x};\quad \text{then}\quad \mathbf{H_x} = \mathbf{I}_4$$

and

$$\begin{bmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\\ \Delta y_4\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\Delta x_1\\ \Delta x_2\\ \Delta x_3\\ \Delta x_4\end{bmatrix} \triangleq \begin{bmatrix}\Delta V\\ \Delta \gamma\\ \Delta q\\ \Delta \alpha\end{bmatrix}$$

Only output is ΔV

$$\Delta y = \Delta V = \begin{bmatrix}1&0&0&0\end{bmatrix}\begin{bmatrix}\Delta V\\ \Delta \gamma\\ \Delta q\\ \Delta \alpha\end{bmatrix}$$

ΔV and $\Delta\alpha$ are measured

$$\Delta\mathbf{y} = \begin{bmatrix}\Delta y_1\\ \Delta y_2\end{bmatrix} = \begin{bmatrix}\Delta V\\ \Delta \alpha\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\Delta V\\ \Delta \gamma\\ \Delta q\\ \Delta \alpha\end{bmatrix}$$

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A Little More About Output Matrices

- Output (measurement) of body-axis velocity and pitch rate and angle
- Transformation from [ΔV, $\Delta\gamma$, Δq, $\Delta\theta$] to [Δu, Δw, Δq, $\Delta\alpha$]

$$\begin{bmatrix}\Delta u\\ \Delta w\\ \Delta q\\ \Delta\theta\end{bmatrix} = \begin{bmatrix}\cos\alpha_N & 0 & 0 & -V_N\sin\alpha_N\\ \sin\alpha_N & 0 & 0 & V_N\cos\alpha_N\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}\Delta V\\ \Delta\gamma\\ \Delta q\\ \Delta\alpha\end{bmatrix}$$

- Separate measurement of state and control perturbations

$$\Delta\mathbf{y} = \begin{bmatrix}\Delta\mathbf{x}\\ \Delta\mathbf{u}\end{bmatrix} = \mathbf{H}_x\Delta\mathbf{x} + \mathbf{H}_u\Delta\mathbf{u}$$

$$\begin{bmatrix}\Delta y_1\\ \Delta y_2\\ \Delta y_3\\ \Delta y_4\\ \Delta y_5\\ \Delta y_6\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\Delta V\\ \Delta\gamma\\ \Delta q\\ \Delta\alpha\end{bmatrix} + \begin{bmatrix}0&0\\0&0\\0&0\\0&0\\1&0\\0&1\end{bmatrix}\begin{bmatrix}\Delta\delta E\\ \Delta\delta T\end{bmatrix}$$

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Simplification of Angle-of-Attack-Rate Effects

Neglecting L_q and $L_{\dot\alpha}$

$$\begin{bmatrix}\Delta\dot q\\ \Delta\dot\alpha\end{bmatrix} \simeq \begin{bmatrix}\{M_q + M_{\dot\alpha}\} & \left\{M_\alpha - M_{\dot\alpha}\left(L_\alpha/V_N\right)\right\}\\ 1 & -\left(L_\alpha/V_N\right)\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta\alpha\end{bmatrix} + \begin{bmatrix}M_{\delta E} - M_{\dot\alpha}\left(L_{\delta E}/V_N\right)\\ -\left(L_{\delta E}/V_N\right)\end{bmatrix}\Delta\delta E$$

Typically

L_q and $L_{\dot\alpha}$ have small effects for large aircraft*

M_q and $M_{\dot\alpha}$ are same order of magnitude and are more significant

* but not for small aircraft, e.g., R/C models and micro-UAVs

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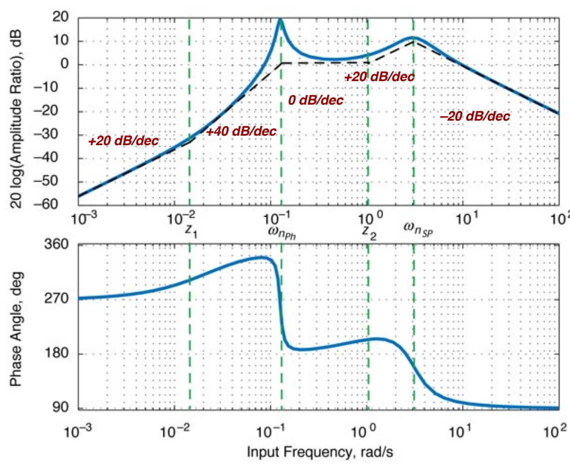
Elevator-to-Pitch-Rate Frequency Response

$$\frac{\Delta q(s)}{\Delta \delta E(s)} = \frac{n_{\delta E}^{q}(s)}{\Delta_{Lon}(s)} \approx \frac{M_{\delta E}\, s\left(s - z_1\right)\left(s - z_2\right)}{\left(s^2 + 2\zeta\omega_n s + \omega_n^{\,2}\right)_{Ph}\left(s^2 + 2\zeta\omega_n s + \omega_n^{\,2}\right)_{SP}}$$

$$\triangleq \frac{M_{\delta E}\, s\left(s + 1/T_{\theta_1}\right)\left(s + 1/T_{\theta_2}\right)}{\left(s^2 + 2\zeta\omega_n s + \omega_n^{\,2}\right)_{Ph}\left(s^2 + 2\zeta\omega_n s + \omega_n^{\,2}\right)_{SP}}$$

- $(n - q) = 1$
- Negligible low-frequency response, except at phugoid natural frequency
- High-frequency response well predicted by 2nd-order model



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Elevator-to-Normal-Velocity Numerator

($L_{\delta E} = 0$)

Transform though α_N back to body axes

$$\mathbf{H_x} Adj\left(s\mathbf{I} - \mathbf{F}_{Lon}\right)\mathbf{G} = \begin{bmatrix} \sin\alpha_N & 0 & 0 & V_N\cos\alpha_N \end{bmatrix} \begin{bmatrix} n_V^V(s) & n_\gamma^V(s) & n_q^V(s) & n_\alpha^V(s) \\ n_V^\gamma(s) & n_\gamma^\gamma(s) & n_q^\gamma(s) & n_\alpha^\gamma(s) \\ n_V^q(s) & n_\gamma^q(s) & n_q^q(s) & n_\alpha^q(s) \\ n_V^\alpha(s) & n_\gamma^\alpha(s) & n_q^\alpha(s) & n_\alpha^\alpha(s) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M_{\delta E} \\ 0 \end{bmatrix} = n_{\delta E}^{w}(s)$$

Scalar transfer function numerator

$$n_{\delta E}^{w}(s) = \begin{bmatrix} \sin\alpha_N & 0 & 0 & V_N\cos\alpha_N \end{bmatrix} \begin{bmatrix} n_q^V(s) \\ n_q^\gamma(s) \\ n_q^q(s) \\ n_q^\alpha(s) \end{bmatrix} M_{\delta E}$$

$$= M_{\delta E}\left[\left(\sin\alpha_N\right)n_q^V(s) + \left(V_N\cos\alpha_N\right)n_q^\alpha(s)\right]$$

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Elevator-to-Normal-Velocity Transfer Function

$$\frac{\Delta w(s)}{\Delta \delta E(s)} = \frac{n_{\delta E}^{w}(s)}{\Delta_{Lon}(s)} = \frac{M_{\delta E}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{Approx\ Ph}\left(s - z_3\right)}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{Ph}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{SP}}$$

- **Normal velocity transfer function is analogous to angle of attack transfer function ($\Delta\alpha \approx \Delta w/V_N$)**
- **z_3 often neglected due to high frequency**

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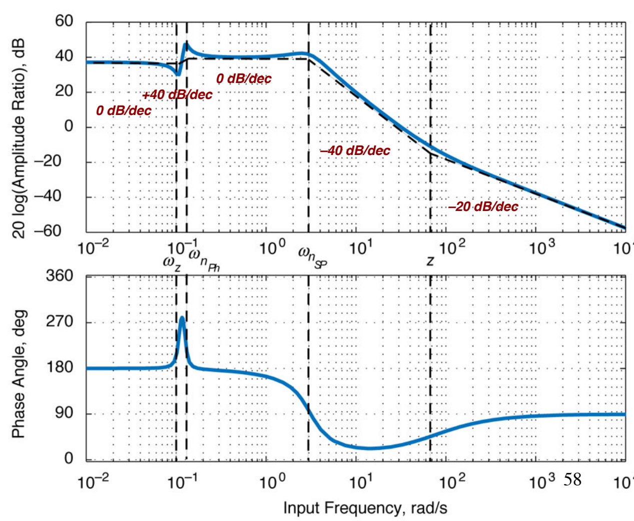
Elevator-to-Normal-Velocity Frequency Response

$$\frac{\Delta w(s)}{\Delta \delta E(s)} = \frac{n_{\delta E}^{w}(s)}{\Delta_{Lon}(s)} \approx \frac{M_{\delta E}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{Approx\ Ph}\left(s - z_3\right)}{\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{Ph}\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)_{SP}}$$

- **$(n - q) = 1$**
- **Complex zero almost (but not quite) cancels phugoid response**



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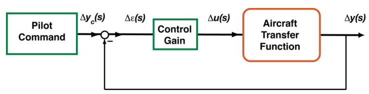
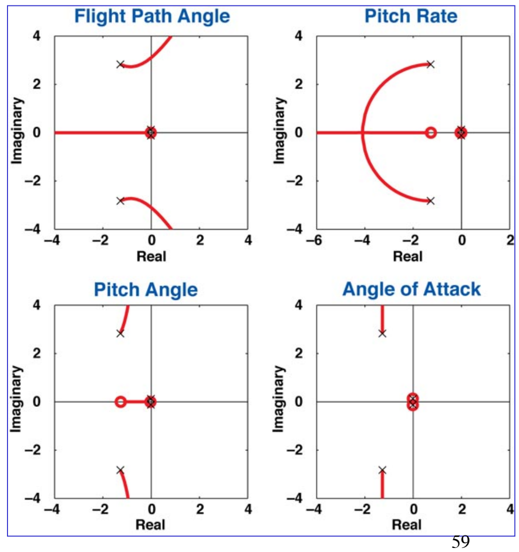
Feedback Control: Angles to Elevator



- **Variations in control gain**
- **Principal effect is on short-period roots**



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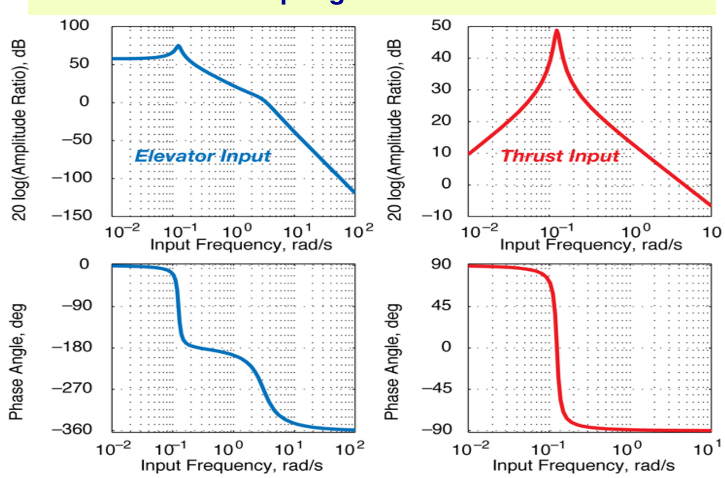
Airspeed Frequency Response to Elevator and Thrust Inputs

Response is primarily through the lightly damped phugoid mode



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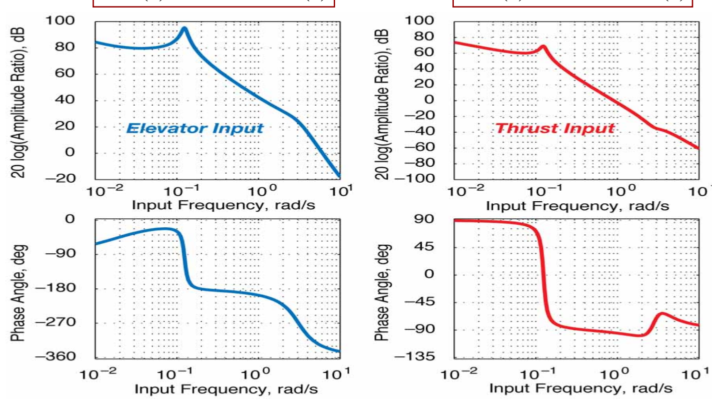
Altitude Frequency Response to Elevator and Thrust Inputs

Altitude perturbation: Integral of the flight path angle perturbation

$$\Delta z(t) = -V_N \int_0^t \Delta\gamma(\tau)\, d\tau$$

$$\frac{\Delta z(s)}{\Delta\delta E(s)} = -\left(\frac{V_N}{s}\right)\frac{\Delta\gamma(s)}{\Delta\delta E(s)} \qquad \frac{\Delta z(s)}{\Delta\delta T(s)} = -\left(\frac{V_N}{s}\right)\frac{\Delta\gamma(s)}{\Delta\delta T(s)}$$



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High- and Low-Frequency Limits of Frequency Response Function

$$\mathscr{H}_{ij}(j\omega) = AR(\omega)\, e^{j\phi(\omega)}$$

$$\mathscr{H}_{ij}(j\omega \to \infty) \to \frac{k_{ij}\left[(j\omega)^q + b_{q-1}(j\omega)^{q-1} + \ldots + b_1(j\omega) + b_0\right]}{\left[(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1(j\omega) + a_0\right]} \to \frac{k_{ij}}{(j\omega)^{n-q}}$$

$$\mathscr{H}_{ij}(j\omega \to 0) \to \frac{k_{ij}\left[(j\omega)^q + b_{q-1}(j\omega)^{q-1} + \ldots + b_1(j\omega) + b_0\right]}{\left[(j\omega)^n + a_{n-1}(j\omega)^{n-1} + \ldots + a_1(j\omega) + a_0\right]} \to \begin{cases} \dfrac{k_{ij} b_0}{a_0}, & b_0 \neq 0 \\[2ex] \dfrac{k_{ij} j(0) b_1}{a_0}, & b_0 = 0, b_1 \neq 0, etc. \end{cases}$$

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Elevator-to-Pitch-Rate Numerator and Transfer Function

$$\mathbf{H}_x Adj\left(s\mathbf{I}-\mathbf{F}_{Lon}\right)\mathbf{G} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} n_V^V(s) & n_\gamma^V(s) & n_q^V(s) & n_\alpha^V(s) \\ n_V^\gamma(s) & n_\gamma^\gamma(s) & n_q^\gamma(s) & n_\alpha^\gamma(s) \\ n_V^q(s) & n_\gamma^q(s) & n_q^q(s) & n_\alpha^q(s) \\ n_V^\alpha(s) & n_V^\alpha(s) & n_V^\alpha(s) & n_V^\alpha(s) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M_{\delta E} \\ 0 \end{bmatrix} = n_{\delta E}^q(s)$$

$$\frac{\Delta q(s)}{\Delta \delta E(s)} = \frac{n_{\delta E}^q(s)}{\Delta_{Lon}(s)} \approx \frac{M_{\delta E}\, s\left(s-z_1\right)\left(s-z_2\right)}{\left(s^2+2\zeta\omega_n s+\omega_n^2\right)_{Ph}\left(s^2+2\zeta\omega_n s+\omega_n^2\right)_{SP}}$$

"Free *s*" in numerator differentiates pitch angle transfer function

Transfer Functions of Thrust Input to Angle Output

Thrust-to-Flight Path Angle transfer function

$$\frac{\Delta\gamma(s)}{\Delta\delta T(s)} = \frac{n_{\delta T}^\gamma(s)}{\Delta_{Lon}(s)}; \quad n_{\delta T}^\gamma(s) = T_{\delta T}\frac{L_V}{V_N}\left(s^2+2\zeta\omega_n s+\omega_n^2\right)_{Approx\ SP}$$

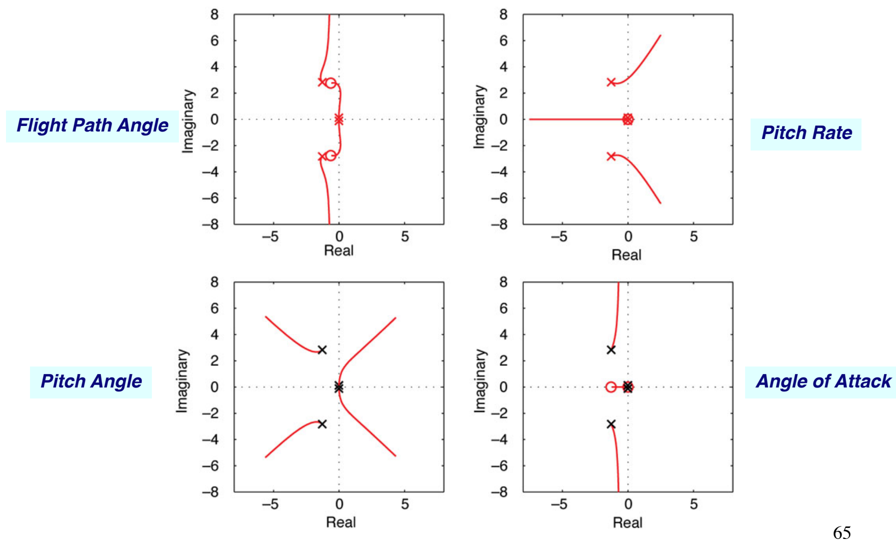
Thrust-to-Angle of Attack transfer function

$$\frac{\Delta\alpha(s)}{\Delta\delta T(s)} = \frac{n_{\delta T}^\alpha(s)}{\Delta_{Lon}(s)}; \quad n_{\delta T}^\alpha(s) = T_{\delta T}\, s\left(s + 1/T_{\alpha_T}\right)$$

Thrust-to-Pitch Angle transfer function

$$\frac{\Delta\theta(s)}{\Delta\delta T(s)} = \frac{n_{\delta T}^\theta(s)}{\Delta_{Lon}(s)}; \quad n_{\delta T}^\theta(s) = T_{\delta T}\left(s + 1/T_{\theta_T}\right)$$

Root Locus Analysis of Angular Feedback to Thrust (4th-Order Model)



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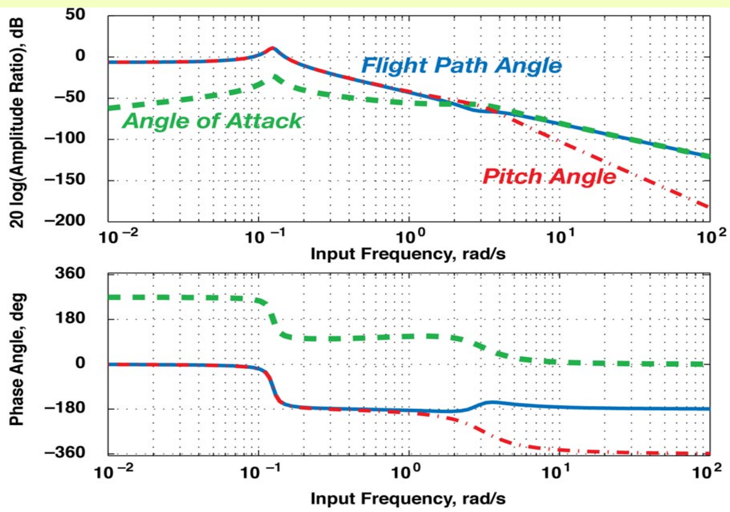
Frequency Response of Angles to Thrust Input

- Primarily effects flight path angle and low-frequency pitch angle



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Angle-of-Attack-Rate Effects

Pre-multiply both sides by inverse

$$\begin{bmatrix}\Delta\dot{q}\\ \Delta\dot{\alpha}\end{bmatrix} = \begin{bmatrix}1 & -M_{\dot{\alpha}}\\ 0 & \left[1+\left(L_{\dot{\alpha}}/V_N\right)\right]\end{bmatrix}^{-1}\left\{\begin{bmatrix}M_q & M_\alpha\\ \left(1-L_q/V_N\right) & -\left(L_\alpha/V_N\right)\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta\alpha\end{bmatrix}+\begin{bmatrix}M_{\delta E}\\ -\left(L_{\delta E}/V_N\right)\end{bmatrix}\Delta\delta E\right\}$$

Inverse of the apparent mass matrix

$$\begin{bmatrix}1 & -M_{\dot{\alpha}}\\ 0 & \left[1+\left(L_{\dot{\alpha}}/V_N\right)\right]\end{bmatrix}^{-1} = \frac{\begin{bmatrix}\left[1+\left(L_{\dot{\alpha}}/V_N\right)\right] & M_{\dot{\alpha}}\\ 0 & 1\end{bmatrix}}{\left[1+\left(L_{\dot{\alpha}}/V_N\right)\right]}$$

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Angle-of-Attack-Rate Effects

Substitute

$$\begin{bmatrix}\Delta\dot{q}\\ \Delta\dot{\alpha}\end{bmatrix} = \frac{\begin{bmatrix}\left[1+\left(L_{\dot{\alpha}}/V_N\right)\right] & M_{\dot{\alpha}}\\ 0 & 1\end{bmatrix}}{\left[1+\left(L_{\dot{\alpha}}/V_N\right)\right]}\left\{\begin{bmatrix}M_q & M_\alpha\\ \left(1-L_q/V_N\right) & -L_\alpha/V_N\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta\alpha\end{bmatrix}+\cdots\right\}$$

Multiply matrices

$$\begin{bmatrix}\Delta\dot{q}\\ \Delta\dot{\alpha}\end{bmatrix} = \frac{1}{\left[1+\left(L_{\dot{\alpha}}/V_N\right)\right]}\left\{\begin{bmatrix}\left\{\left[1+\left(L_{\dot{\alpha}}/V_N\right)\right]M_q + M_{\dot{\alpha}}\left(1-L_q/V_N\right)\right\} & \left\{\left[1+\left(L_{\dot{\alpha}}/V_N\right)\right]M_\alpha - M_{\dot{\alpha}}\left(L_\alpha/V_N\right)\right\}\\ \left(1-L_q/V_N\right) & -L_\alpha/V_N\end{bmatrix}\begin{bmatrix}\Delta q\\ \Delta\alpha\end{bmatrix} + \begin{bmatrix}\left[1+\left(L_{\dot{\alpha}}/V_N\right)\right]M_{\delta E} - M_{\dot{\alpha}}\left(L_{\delta E}/V_N\right)\\ -L_{\delta E}/V_N\end{bmatrix}\Delta\delta E\right\}$$

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Power Effects on 4th-Order Longitudinal Modes

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Power Effects on Stability and Control

- *Gee Bee R1 Racer*: an engine with wings and almost no tail
- During W.W.II, the size of fighters remained about the same, but installed horsepower doubled (*F4F* vs. *F8F*)
- Use of flaps means high power at low speed, increasing relative significance of thrust effects
- Short-Takeoff-and-Landing (STOL) aircraft augment takeoff/landing lift in many ways, e.g.,
  - Full-span flaps
  - Deflected thrust

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Direct Thrust Effect on Speed Stability, T_V

- In steady, level flight, nominal thrust balances nominal drag

$$T_N - D_N = C_{T_N}\frac{1}{2}\rho V_N^2 S - C_{D_N}\frac{1}{2}\rho V_N^2 S = 0$$

- Effect of velocity change

$$\frac{\partial T}{\partial V} = \begin{cases} < 0, & \textit{for propeller aircraft} \\ \approx 0, & \textit{for turbojet aircraft} \\ > 0, & \textit{for ramjet aircraft} \end{cases}$$

- Small velocity perturbation grows if
- Therefore
  - propeller is stabilizing for velocity change
  - turbojet has neutral effect
  - ramjet is destabilizing

$$\frac{\partial T}{\partial V} - \frac{\partial D}{\partial V} > 0$$

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Steady-State Response of the 4th-Order LTI Longitudinal Model

$$\Delta\dot{\mathbf{x}}(t) = \mathbf{F}\Delta\mathbf{x}(t) + \mathbf{G}\Delta\mathbf{u}(t)$$

- How do we calculate the equilibrium response to control?

$$\Delta\mathbf{x}_{SS} = -\mathbf{F}^{-1}\mathbf{G}\,\Delta\mathbf{u}_{SS}$$

- For the longitudinal model

$$\begin{bmatrix} \Delta V_{SS} \\ \Delta\gamma_{SS} \\ \Delta q_{SS} \\ \Delta\alpha_{SS} \end{bmatrix} = -\left[\begin{array}{cc|cc} -D_V & -g & 0 & -D_\alpha \\ L_V/V_N & 0 & 0 & L_\alpha/V_N \\ \hline M_V & 0 & M_q & M_\alpha \\ -L_V/V_N & 0 & 1 & -L_\alpha/V_N \end{array}\right]^{-1} \begin{bmatrix} 0 & T_{\delta T} & 0 \\ 0 & 0 & L_{\delta F}/V_N \\ M_{\delta E} & 0 & 0 \\ 0 & 0 & -L_{\delta F}/V_N \end{bmatrix} \begin{bmatrix} \Delta\delta E_{SS} \\ \Delta\delta T_{SS} \\ \Delta\delta F_{SS} \end{bmatrix}$$

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Algebraic Equation for Equilibrium Response

$$\begin{bmatrix}\Delta V_{SS}\\ \Delta\gamma_{SS}\\ \Delta q_{SS}\\ \Delta\alpha_{SS}\end{bmatrix} = \frac{\begin{bmatrix} \left[-gM_{\delta E}\frac{L_\alpha}{V_N}\right] & 0 & \left[gM_\alpha L_{\delta F}/V_N\right] \\ \left[\left(D_V\frac{L_\alpha}{V_N}-D_\alpha\frac{L_V}{V_N}\right)M_{\delta E}\right] & \left[\left(M_V\frac{L_\alpha}{V_N}-M_\alpha\frac{L_V}{V_N}\right)T_{\delta T}\right] & \left[\left(D_\alpha M_V - D_V M_\alpha\right)L_{\delta F}/V_N\right] \\ 0 & 0 & 0 \\ \left[-gM_{\delta E}\frac{L_V}{V_N}\right] & 0 & \left[L_{\delta F}/V_N\right] \end{bmatrix}}{g\left(M_V\frac{L_\alpha}{V_N}-M_\alpha\frac{L_V}{V_N}\right)} \begin{bmatrix}\Delta\delta E_{SS}\\ \Delta\delta T_{SS}\\ \Delta\delta F_{SS}\end{bmatrix}$$

- **Roles of stability and control derivatives identified**
- **Result is a simple equation relating input and output**

$$\begin{bmatrix}\Delta V_{SS}\\ \Delta\gamma_{SS}\\ \Delta q_{SS}\\ \Delta\alpha_{SS}\end{bmatrix} = \begin{bmatrix} a & 0 & b \\ c & d & e \\ 0 & 0 & 0 \\ f & 0 & g \end{bmatrix} \begin{bmatrix}\Delta\delta E_{SS}\\ \Delta\delta T_{SS}\\ \Delta\delta F_{SS}\end{bmatrix}$$

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4th-Order Steady-State Response May Be Counterintuitive

$$\Delta V_{SS} = a\Delta\delta E_{SS} + (0)\Delta\delta T_{SS} + b\Delta\delta F_{SS}$$
$$\Delta\gamma_{SS} = c\Delta\delta E_{SS} + d\Delta\delta T_{SS} + e\Delta\delta F_{SS}$$
$$\Delta q_{SS} = (0)\Delta E_{SS} + (0)\Delta\delta T_{SS} + (0)\Delta\delta F_{SS}$$
$$\Delta\alpha_{SS} = f\Delta\delta E_{SS} + (0)\Delta\delta T_{SS} + g\Delta\delta F_{SS}$$

- **Observations**
  - **Thrust command**
  - **Elevator and flap commands**
  - **Steady-state pitch rate is zero**
  - ***4th-order model neglects air density gradient effects***

Steady-state pitch angle

$$\Delta\theta_{SS} = \Delta\gamma_{SS} + \Delta\alpha_{SS} = (c+f)\Delta\delta E_{SS} + d\Delta\delta T_{SS} + (e+g)\Delta\delta F_{SS}$$

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Examples of Gain and Phase Margins: 2nd-Order System with Low-Pass Filter

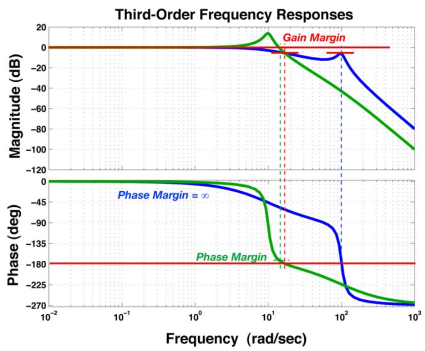
Low-Bandwidth Filter

$$\mathscr{H}_{blue}(j\omega)=\left[\frac{10}{(j\omega+10)}\right]\left[\frac{100^2}{(j\omega)^2+2(0.1)(100)(j\omega)+100^2}\right]$$

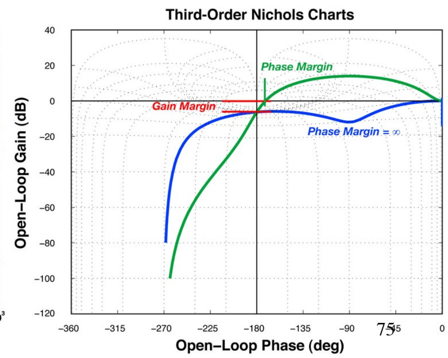
High-Bandwidth Filter

$$\mathscr{H}_{green}(j\omega)=\left[\frac{10^2}{(j\omega)^2+2(0.1)(10)(j\omega)+10^2}\right]\left[\frac{100}{(j\omega+100)}\right]$$

Bode Plot



Nichols Chart



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