Formal Logic, Algorithms, and Incompleteness

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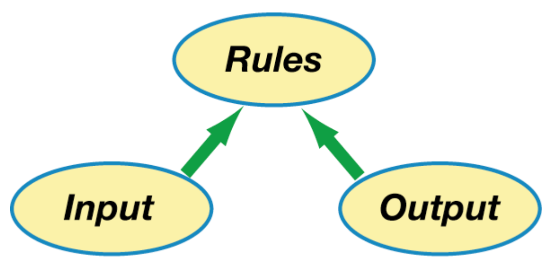
Robotics and Intelligent Systems MAE 345,
Princeton University, 2017

Learning Objectives

- **Principles of axiomatic systems and formal logic**
- **Application of logic in computing machines**
- **Algorithms and numbering systems**
- **Gödel's Theorems: What axiomatic systems can't do**

X	Y	$X \wedge Y$	$X \vee Y$	$X \to Y$	$X \equiv Y$	$X \wedge (\neg Y)$	…
T	T	T	T	T	T	F	
T	F	F	T	F	F	T	
F	T	F	T	T	F	F	
F	F	F	F	T	T	F	



Intelligent Systems

- **Perform useful functions driven by desired goals and current knowledge**
  - **Emulate biological and cognitive processes**
  - **Process information to achieve objectives**
  - **Learn by example or from experience**
  - **Adapt functions to a changing environment**

Should robots be "More like us?"

- **Semantics: The study of meaning**
- **Syntax: Orderly or systematic arrangement of parts or elements**

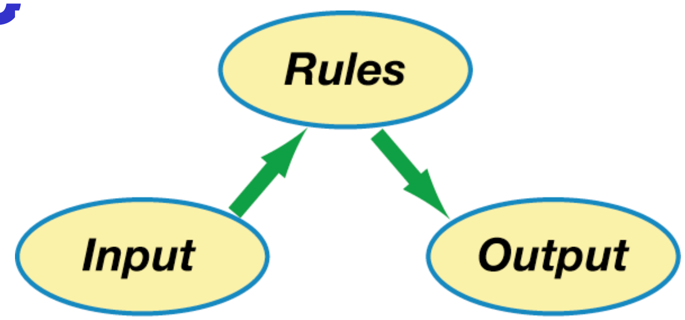
Cognitive Paradigms for Intelligent Systems

- **Thinking**
  - Syntax
  - Algorithmic behavior
  - Comparison
  - Reflection
- **Consciousness**
  - Understanding and judgment of truth
- **Intelligence**
  - Flexible response
  - Recognition of similarity and contradiction
  - Ranking of information
  - Synthesis of solutions
  - Reasoning

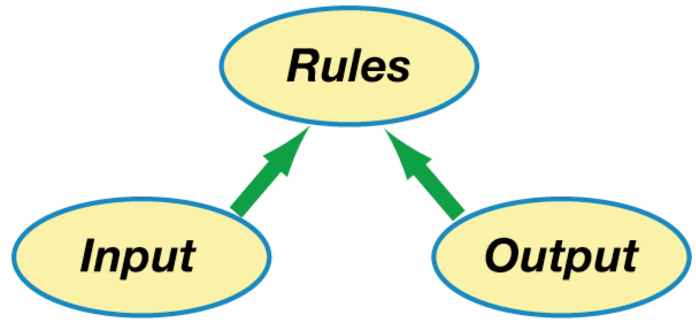
Underlying structure: Logic

Formal Logic

- **Deduction**
  - Shows that a proposition follows from one or more other propositions
  - Establishes the validity of a claim or argument
  - Reasons from input to rules to output



- **Induction**
  - Infers a general law or principle from the observation of particular instances
  - Reasons from input and output to rules



- **Inference**
  - Derivation of conclusions from information, as by
    - Deduction
    - Induction
  - Reasoning from something known or assumed, as by
    - Application of rules or meta-rules (i.e., rules about rules)
    - Probability and statistics

"Forms of Inference" Lead to "Formulas"

- **Formulas**
  - Symbols
  - Operations
  - Rules
- **Axioms**
  - Unproved but assumed formulas
  - Starting point for proofs of formulas
- **Theorems**
  - Formulas proved to be true based on
    - Axioms
    - Other theorems
- **Algorithms**
  - Systematic procedures for using formulas
- **Calculus**
  - A system or method of calculation
  - A method of assessment

Propositional Calculus - 1

- **Proposition:** A statement that may be either true or false
- Complete, unanalyzed propositions and combinations
  - What can be said -- formal relations and implications -- axioms of the system
  - Deductive structure: Rules of Inference
  - Concern with form or syntax of statements
  - Meaning of a statement may not be self-evident; for example,

$$(2+3), \quad (+\,2\;3), \quad (2\;3\,+)$$

  - may be different notations for the same statement

Infix	Prefix	Postfix
"Algebraic notation"	*"Reverse Polish notation"*	*"Polish notation"*
?	1954	1924

Examples of Propositions

Princeton's colors are orange and black (true) … are red and gray (false)

6 + 6 = 12; 6 + 7 = 12

"I have a bridge to sell to you …."

Variables and Operators
(or *Sentential Variables and Connectives*)

- Sentential variables may be either true or false
- Operators connect sentential (or propositional) variables
- A proposition (or sentence) is a formula containing variables and operators

And	$\wedge$ or &	*Conjunction*
Or	$\vee$	*Disjunction*
Not	$\neg$ or $\sim$	*Negation*
Implies	$\rightarrow$ or $\supset$	*Material Implication (If)*
Equivalent	$\equiv$ or $\leftrightarrow$	*Material Equivalence (If and only if)*

Dyadic Operations - 1

- **Operations involving two arguments (i.e., sentential variables)**
- **Arguments of operators = Propositions**
  - ***X* represents "Socrates is a man"**
  - ***Y* represents "All men are mortal"**
- **Examples of formulas or connective expressions [dyadic operations (2 arguments)]**

$$X \wedge Y$$

$$X \vee Y$$

- **"Socrates is a man" and "All men are mortal"**
- **"Socrates is a man" or "All men are mortal"**

Dyadic Operations - 2

$X \to Y$

$X \equiv Y$

- "Socrates is a man" implies that "All men are mortal"
- "Socrates is a man" is equivalent to "All men are mortal"

- 1st argument is the antecedent; 2nd argument is the consequent

- "IF-THEN-ELSE" interpretation of dyadic operations
  - If *X* is true and *Y* is true, then $X \wedge Y$ is true; else $X \wedge Y$ is false
  - If *X* is true or *Y* is true, then $X \vee Y$ is true; else $X \vee Y$ is false

Monadic Operations and Syntactic Propositions

- Negation is a monadic (single argument) operation
  - If X is true, then $\neg X$ is false
  - If X is false, then $\neg X$ is true
- Brackets group propositions to form Syntactic Propositions (i.e., propositions based on propositions)
- Incorporation of negation in dyadic operations:

$$X \wedge (\neg Y)$$

$$X \vee (\neg Y)$$

If X is true and Y is false, then $X \wedge (\neg Y)$ is true; else …

If X is true or Y is false, then …

Truth Tables for Dyadic Expressions

X	Y	$X \wedge Y$	$X \vee Y$	$X \rightarrow Y$	$X \equiv Y$	$X \wedge (\neg Y)$	…
T	T	T	T	T	T	F	
T	F	F	T	F	F	T	
F	T	F	T	T	F	F	
F	F	F	F	T	T	F	

- **Syntactic combinations build sentences**
- **Tautology (repetitive statement) is always true**
  - **"*X* implies *Y* and *Z*" is the same as "*X* implies *Y* and *X* implies *Z*"**

$$\left(X \rightarrow \left(Y \wedge Z\right)\right) \equiv \left(\left(X \rightarrow Y\right) \wedge \left(X \rightarrow Z\right)\right)$$

More Concepts in Propositional Calculus

- **Fallacy or Contradiction**
  - **Saying that [*X* or *Y* is false is the same as saying that "*X* is false and *Y* is false" is false)] is a fallacy or contradiction**

$$\neg(X \vee Y) \equiv \neg(\neg Y \wedge \neg X)$$

  - **<u>Liar's paradox</u>: "I am lying." True or false? Sentence refers to its own truth.**
- **Truth depends on the propositions described by *X*, *Y*, and *Z***

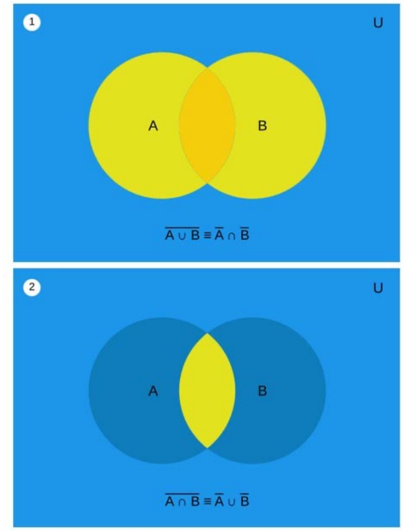
$$(X \wedge Y) \vee (\neg Y \wedge Z)$$

- **Well-formed formulas (*WFFs*) make sense and are unambiguous**

$$(X \wedge Y) \vee (\neg YY(Z)) \quad \text{Not a } \mathit{WFF}$$

More Concepts in Propositional Calculus

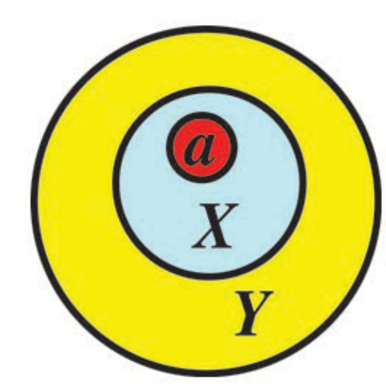
- **Decisions are based on testing the validity of *WFFs***
- **De Morgan's Laws**
  - **Two propositions are jointly true only if neither is false**



$$\neg(X \wedge Y) \equiv \neg X \vee \neg Y$$

$$\neg(X \vee Y) \equiv \neg X \wedge \neg Y$$

- ***Modus Ponens* rule (rule of detachment or elimination)**
  - **If *X* is true and *X* implies *Y*, then we can infer that *Y* is true**



$$(X \wedge (X \rightarrow Y)) \rightarrow Y$$

Modus Ponens Rule

- **Rule of detachment, elimination, definition, or substitution**
  - **If *X* is true and *X* implies *Y*, then we can infer that *Y* is true**

$$(X \wedge (X \rightarrow Y)) \rightarrow Y$$

  - ***X* is true and *X* implies *Y*, then (*X* is true and *X* implies *Y)* implies that *Y* is true**
- **Example from Wikipedia:**
  - **If it's raining, I'll meet you at the movie theater.**
  - **It's raining.**
  - **Therefore, I'll meet you at the movie theater**

Material Implication

- *X* -> *Y*
- Same as "¬*X* or *Y*"
- *X* is false does not imply that *Y* is not true
- "If", not "If and only if", which is material equivalency
- Double negative

- Example:
  - *X*: Anyone can be caught in the rain
  - *Y*: That person is wet
  - *X* -> *Y*, or (if *X Y*)
  - Suppose Dave is wet; was he caught in the rain?
  - Dave went under a sprinkler and got wet; he was not caught in the rain, but he is wet
  - Therefore [(false) -> (true)] is true
  - Material implication does not indicate causality

Material Implication *(if)* vs. Material Equivalence *(iff)*

- $X \equiv Y$
- **"If and only if": *iff***
- **The truth of *X* requires the truth of *Y***
- **If: I will eat lunch if the E-Quad Café has tuna salad**
- **Iff: I will eat lunch if and only if the E-Quad Café has tuna salad**

Toward Predicate Calculus

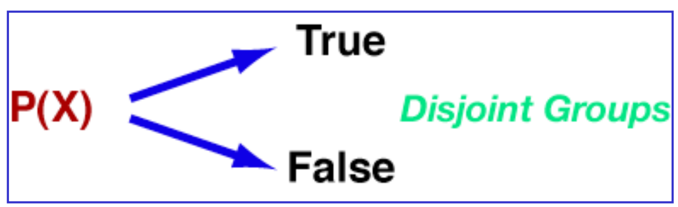
- **Sentence**
  - Series of words forming a grammatically complete expression of a single thought
  - Normally contains (at least) a subject and a predicate
- **Predicate**
  - That which is predicated (or said) of the subject in a proposition
  - Second term of a proposition, e.g.,
    - Socrates is a man
  - The statement made about the subject, e.g.,
    - The main verb, its object, and modifiers

Predicate Calculus

- **Extensions to propositional calculus**
  - **Predicates**
  - **Flexible variables, i.e., more states than only true or false**
  - **Quantification**
    - **Conversion of words to numbers**
    - **Introduction of degrees of value**
  - **Inference rules for quantifiers**
    - **First-order logic**
    - **Productive use of predicates, variables, and quantification**
- **Building blocks for expert systems**

Predicates

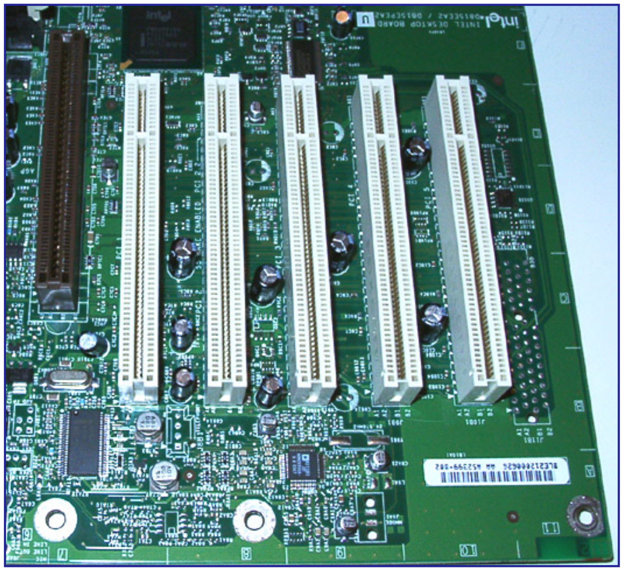
- **Predicate, P(X)**
  - **A statement (or proposition) about individuals (or arguments) that is either true or false***
  - **One argument:**
    Example: "is-red"
  - **Two arguments:**
    Example: "is-greater-than"
  - **QUEEN OF HEARTS is-red (true)**
  - **LIVE GRASS is-red (false)**
  - **SEVEN is-greater-than FOUR**
- **One-argument predicate, P(X), performs a sort**

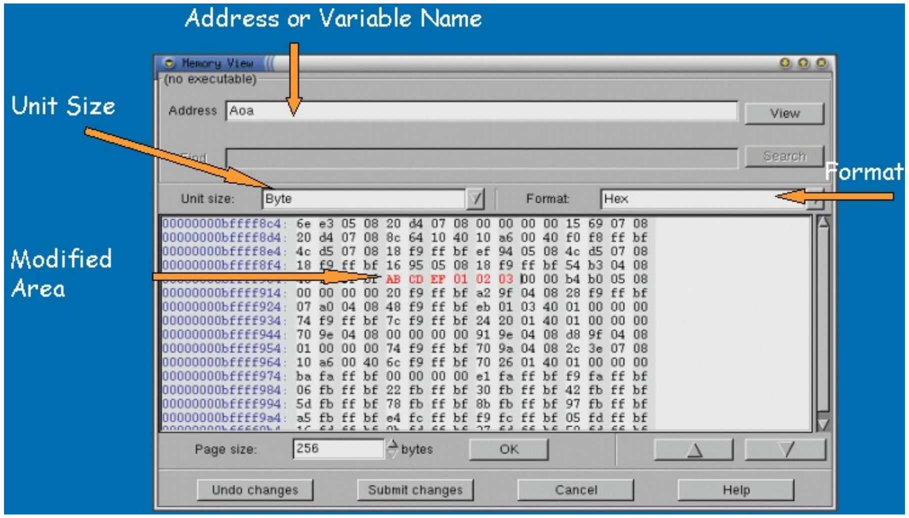


*** also called an atomic formula**

Variable

- A **placeholder** that is to be filled with a **constant**, e.g., **X** in **P(X)**
- A **slot** that receives a **value**
- A **symbolic address** for **information**





Quantification

- "**Universal quantifiers** say something that is true for all possible values of a variable."*

$$\left(forall\ (x)\ f \right)$$

x : variable

f : formula; specifies scope of x

$$\left(forall\ (x) \left(if\ (inst\ x\ fire-engine) (color\ x\ red) \right) \right)$$

- **Existential quantifiers**
  - state conditions under which a variable exists
  - predicate properties or relationships of one or more variables

$$\left(exists\ (x)\ f \right)$$

$$\left(forall\ (x) \left(if\ (person\ x) \left(exists\ (y) (head-of\ x\ y) \right) \right) \right)$$

*** Charniak and McDermott, 1985**

Inference Rules for Quantifiers

- Well-formed formula (WFF)
  - Syntactically correct combination of connectives, predicates, constants, variables, and quantifiers
- Universal Quantification (or Elimination or Instantiation)
  - Man(Socrates) -> Mortal(Socrates)
  - or "The man, Socrates, is mortal" ["given any", "for all"]
- Existential Quantification (or Elimination or Instantiation)
  - Man(person) -> Happy(person)
  - Someone is happy ["there exists at least one"]
- Existential Introduction (Generalization)
  - Man(Jerry) -> Likes_ice_cream(Jerry)
  - Someone likes ice cream ["general to specific" or v.v.]

Examples of Sentences

- LISP-like terms and prefix notation
  - (catch-object jack-1 block-1)
  - (inst block-1 block)
  - (color block-1 blue)
- Jack-1 catches the object called Block-1
- Block-1 is an instantiation of a block
- Block-1 is blue
- With connectives
  - (and (color block-1 yellow) (inst block-1 elephant))
  - (if (supports block-2 block-1) (on block-1 block-2))
  - (if (and (inst clyde elephant) (color elephant gray)) (color clyde gray))
- Block-1 is a yellow elephant
- If block-2 supports block-1, then block-1 is on block-2
- If clyde is an elephant and an elephant is gray, then clyde is gray

First-Order Logic

- Further extensions to predicate calculus
- Functions
  - Fixed number of arguments
  - Rather than returning TRUE or FALSE, functions return objects, e.g.,
    - "uncle-of" Mary returns John
  - Functions of functions, e.g.,
    - (father-of (father-of (John)) returns John's paternal grandfather

First-Order Logic

- **Equals**
  - **Two individuals are equal if and only if (equivalence) they are indistinguishable under all predicates and functions**

$$X \equiv Y$$

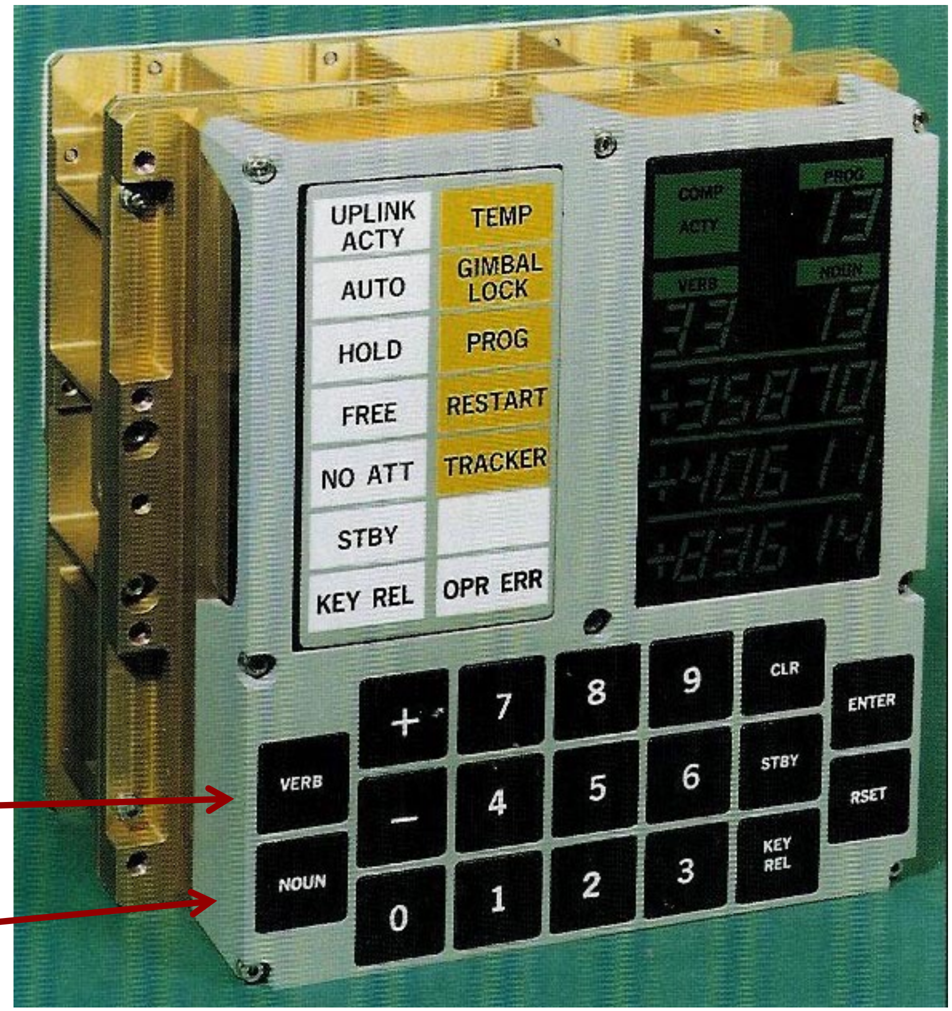
if and only if

$$P(X) \equiv P(Y), \quad F(X) \equiv F(Y), \quad \forall P \wedge F$$

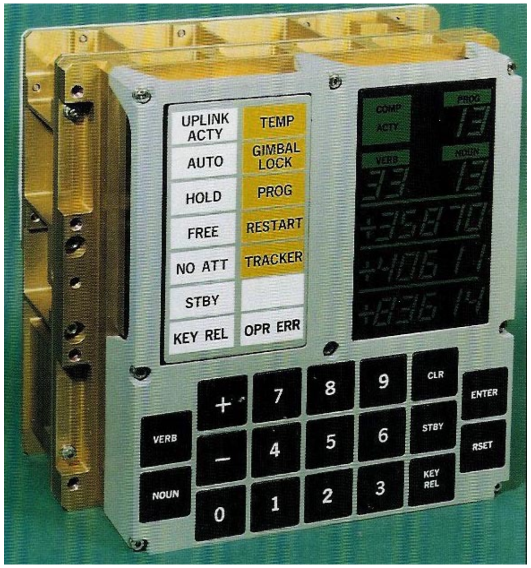
- **Axiomatization**
  - **Axioms: necessary relationships between objects in a domain**
  - **Formal expression in sentences of first-order logic (emphasis on syntax over semantics)**

Apollo Guidance Computer Commands

- **Display/Keyboard (DSKY)**
- **Sentence**
  - **Subject and predicate**
  - **Subject is implied**
    - **Astronaut, or**
    - **GNC system**
  - **Sentence describes action to be taken employing or involving an object**
- **Predicate**
  - **Verb + Noun**
  - **Verb = Action**
  - **Noun = Variable or Program (i.e., the object)**



See *http://www.ibiblio.org/apollo/* for simulation



Numerical Codes for Verbs and Nouns in Apollo Guidance Computer Programs

Verb Code	Description	Remarks
01	Display 1st component of	Octal display of data on REGISTER 1
02	Display 2nd component of	Octal display of data on REGISTER 1
03	Display 3rd component of	Octal display of data on REGISTER 1

Noun Code	Description	Scale/Units
01	Specify machine address	XXXXX
02	Specify machine address	XXXXX
03	(Spare)	
04	(Spare)	
05	Angular error	XXX.XX degrees
06	Pitch angle	XXX.XX degrees
	Heads up-down	+/- 00001
07	Change of program or major mode	
11	Engine ON enable	

Verbs and Nouns in Apollo Guidance Computer Programs

- **Verbs** *(Actions)*
  - Display
  - Enter
  - Monitor
  - Write
  - Terminate
  - Start
  - Change
  - Align
  - Lock
  - Set
  - Return
  - Test
  - Calculate
  - Update

- **Selected Nouns** *(Variables)*
  - Checklist
  - Self-test ON/OFF
  - Star number
  - Failure register code
  - Event time
  - Inertial velocity
  - Altitude
  - Latitude
  - Miss distance
  - Delta time of burn
  - Velocity to be gained

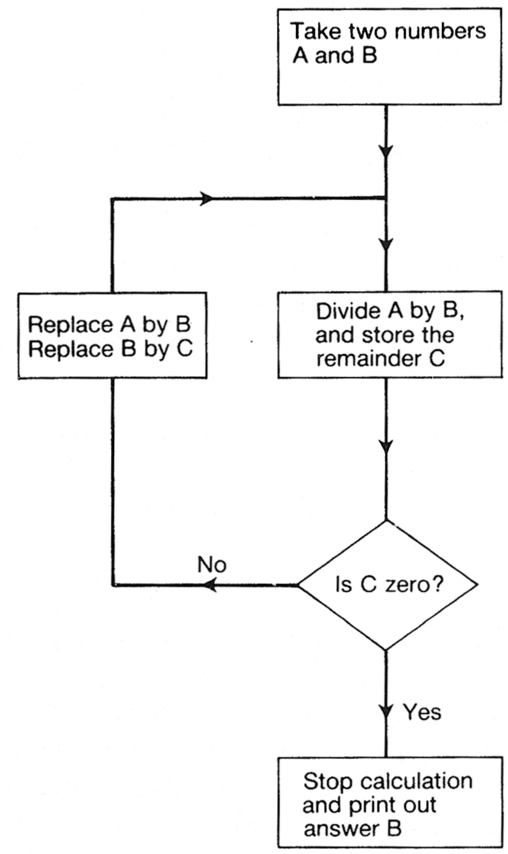
- **Selected Programs (CM)**
  - AGC Idling
  - Gyro Compassing
  - LET Abort
  - Landmark Tracking
  - Ground Track Determination
  - Return to Earth
  - SPS Minimum Impulse
  - CSM/IMU Align
  - Final Phase
  - First Abort Burn

Algorithms

- Systematic procedures for using formulas
- Computer programs contain algorithms
- Euclid's Algorithm
  - Highest common denominator (HCD) of 2 numbers
  - In example, HCD = 21
  - Operations based on natural numbers (positive integers)
- Procedure is completed in a finite number of steps

$3654 \div 1365$ gives remainder 924
$1365 \div 924$ gives remainder 441
$924 \div 441$ gives remainder 42
$441 \div 42$ gives remainder 21
$42 \div 21$ gives remainder 0.

- Flow charts
  - Operations
  - Conditions
  - Sub-routines



Some Natural Numbering Systems

Natural numbers: non-negative, whole numbers

Denary (*Base 10*)	Binary (*Base 2*)	Unary (*Base 1*)
0	0	?
1	1	1
2	10	11
3	11	111
4	100	1111
5	101	11111
6	110	111111
7	111	1111111
8	1000	11111111
9	1001	111111111
10	1010	1111111111
11	1011	11111111111
Digits	Binary Digits "Bits" (John Tukey)	Marks
Two 5-finger hands One 10-finger hand	True-False Yes-No Present-Absent	Chalk and a rock Abacus "Chisenbop"

- Other number systems
  - DNA (*Base 4*) *[ATCG]*
  - Octal (*Base 8*)
  - Hexadecimal (*Base 16*)

$$F3 \\ = (15 \times 16^1) + (3 \times 16^0) \\ = 243$$

Algorithms are Independent of Numbering System

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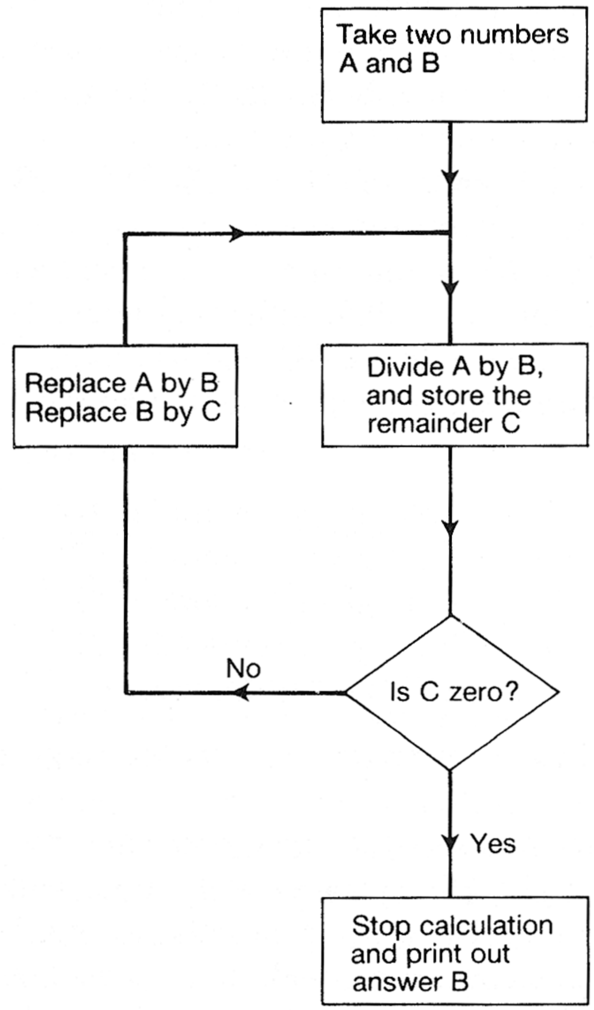
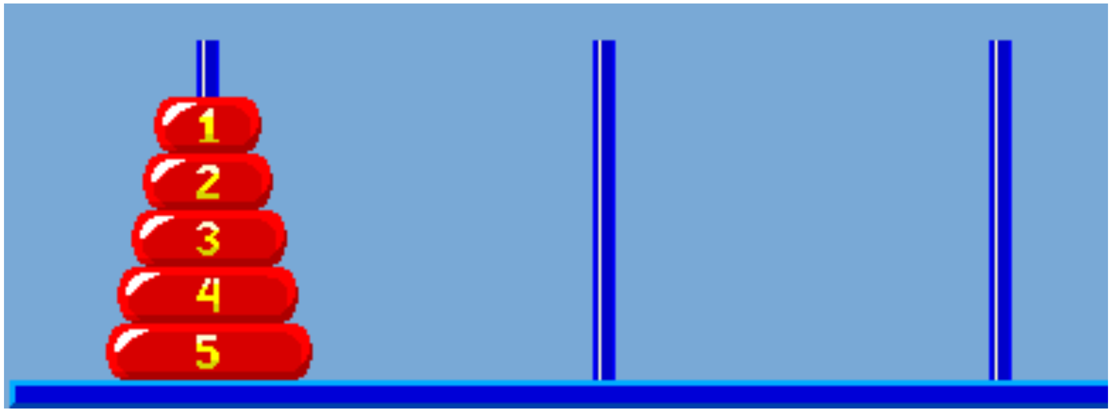
?



- Logical algorithms may deal with objects or symbols directly
- For computation, objects or symbols ultimately are represented by numbers (e.g., 0s and 1s) or alphabet
- Mathematical logical algorithms are independent of the numbering system

Towers of Hanoi:
An Axiomatic System



Problem: **Move all disks (one at a time) from 1st peg to 3rd peg without putting a larger disk on a smaller disk**

- **Objects**
  - **Disks: 1, 2, 3, 4, 5**
  - **Pegs: A, B, C**
- **Predicates**
  - **Sorting: DISK, PEG**
    - **DISK(A) is FALSE**
    - **PEG(A) is TRUE**
  - **Comparison: SMALLER**
    - **SMALLER(1,2) is TRUE**

Barr and Feigenbaum, 1982

Towers of Hanoi

- First axiom

$$\forall XYZ.(\text{SMALLER}(X,Y) \wedge (\text{SMALLER}(Y,Z)) \rightarrow \text{SMALLER}(X,Z)$$

- Premise

$$\text{SMALLER}(1,2) \wedge \text{SMALLER}(2,3)$$

- Situational constant, *S*
  - **Identifies state of system after a series of moves**
- More predicates
  - Vertical relationship: ON
    - **ON(*X*,*Y*,*S*) asserts that disc *X* is on disk *Y* in situation *S***
  - Nothing on top of disk: FREE
    - **FREE*(X*,*S*) indicates that no disc is on *X***

Towers of Hanoi

- **Second axiom***

$$\forall\ X\ S.\mathrm{FREE}(X,S) \equiv \neg\exists\ Y.\left(\mathrm{ON}\left(Y,X,S\right)\right)$$

* "For all disks *X* and situation *S*, *X* is free in situation *S* if and only if there does not exist a disk *Y* such that *Y* is ON *X* in situation *S*."

- **More Predicates**
  - **Acceptable (legal) move: LEGAL (*X*, *Y*, *S*)**
  - **Act of moving disk: MOVE(*X*, *Y*, *S*)**
- **Object of analysis**
  - **Find a situation that is TRUE if a move is legal and is accomplished**
- **More Axioms**
  - **See *Handbook of AI* for additional steps**

Example of theorem proving, i.e., of theory that a goal state can be reached

Gödel's Incompleteness Theorems (1931)

http://en.wikipedia.org/wiki/Gödel's_incompleteness_theorems

- **1st Theorem: "**No consistent system of axioms whose theorems can be listed by an 'effective procedure' (e.g., a computer program …) is capable of proving all truths about the relations of the natural numbers (arithmetic)."
  - "There will always be statements about the natural numbers that are true, but that are unprovable within the system."
- **2nd Theorem:** "Such a system cannot demonstrate its own consistency."
- **~ "Liar's Paradox", replacing "provability" for "truth"**

http://mathworld.wolfram.com/GoedelsIncompletenessTheorem.html

- **1st Theorem:** " Informally, Gödel's incompleteness theorem states that all consistent axiomatic formulations of number theory include undecidable propositions (Hofstadter 1989)."
- **2nd Theorem:** "If number theory is consistent, then a proof of this fact does not exist using the methods of first-order predicate calculus."

Thomas Kuhn: *The Structure of Scientific Revolutions*, 1962

- **Advances in Science**
  - **Not a steady, cumulative acquisition of knowledge**
  - **Peaceful interludes punctuated by intellectually violent revolutions**
- **Paradigm**
  - **Pre-Kuhn: A pattern, exemplar, or example *(OED, 1483)***
  - **Post-Kuhn: "A collection of procedures or ideas that instruct scientists, implicitly, what to believe and how to work." *(Horgan, 2012)***
- **Paradigm Shift**
  - **One world view is replaced by another**
  - **Gödel's theorem: for any axiomatic system there exist propositions that are either undecidable or not provably consistent**
  - **Theory rests on subjective framework**
  - **Propositions are true or false only within the context of a paradigm**

http://blogs.scientificamerican.com/cross-check/2012/05/23/what-thomas-kuhn-really-thought-about-scientific-truth/

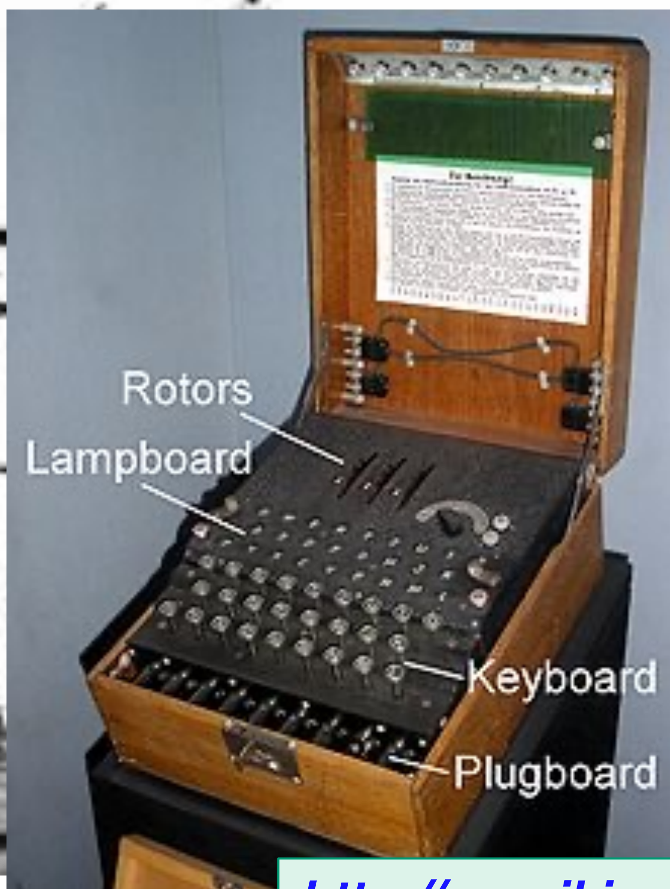
Next Time:
Computers, Computing, and Sets

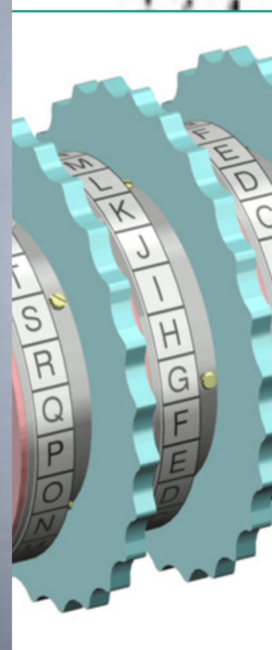
Supplemental Material

Enigma and the Bletchley Park Bombe

26-letter, 3- or 4-rotor encryption device used by German military during WWII

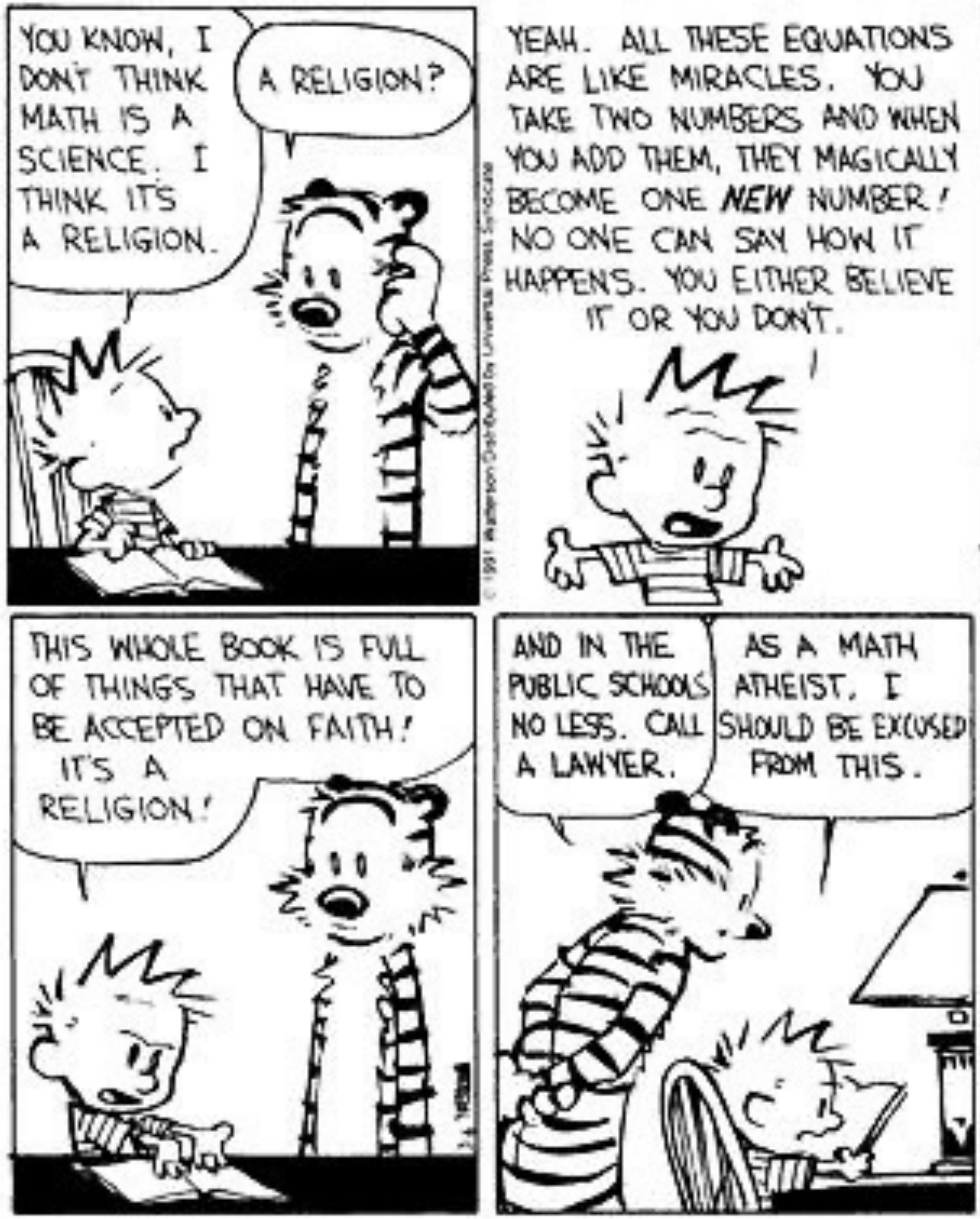
Algorithmic decyphering computer designed by Polish mathematicians, Alan Turing, and US Navy





http://en.wikipedia.org/wiki/Bombe

Calvin and Hobbes

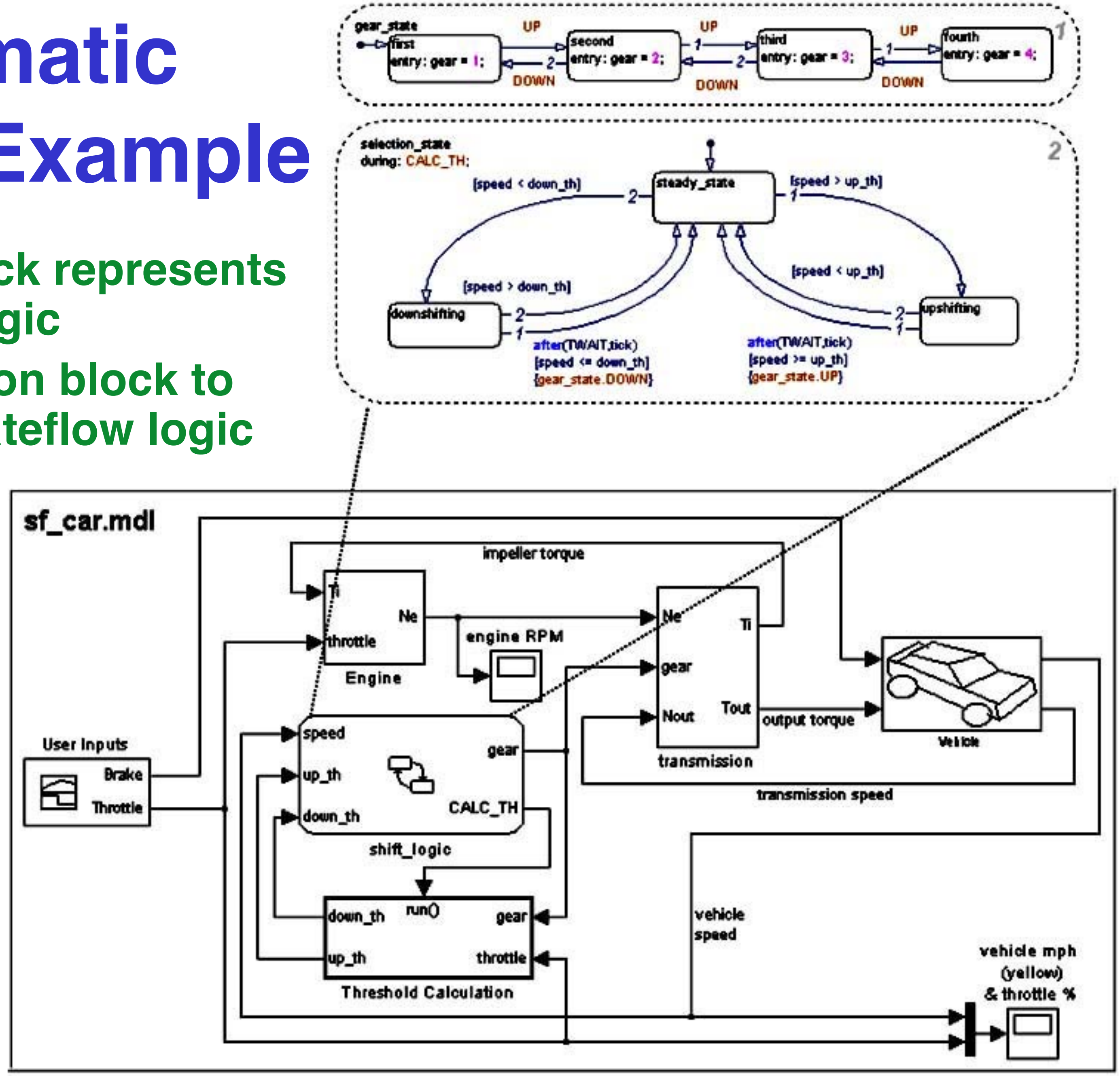


MATLAB Stateflow

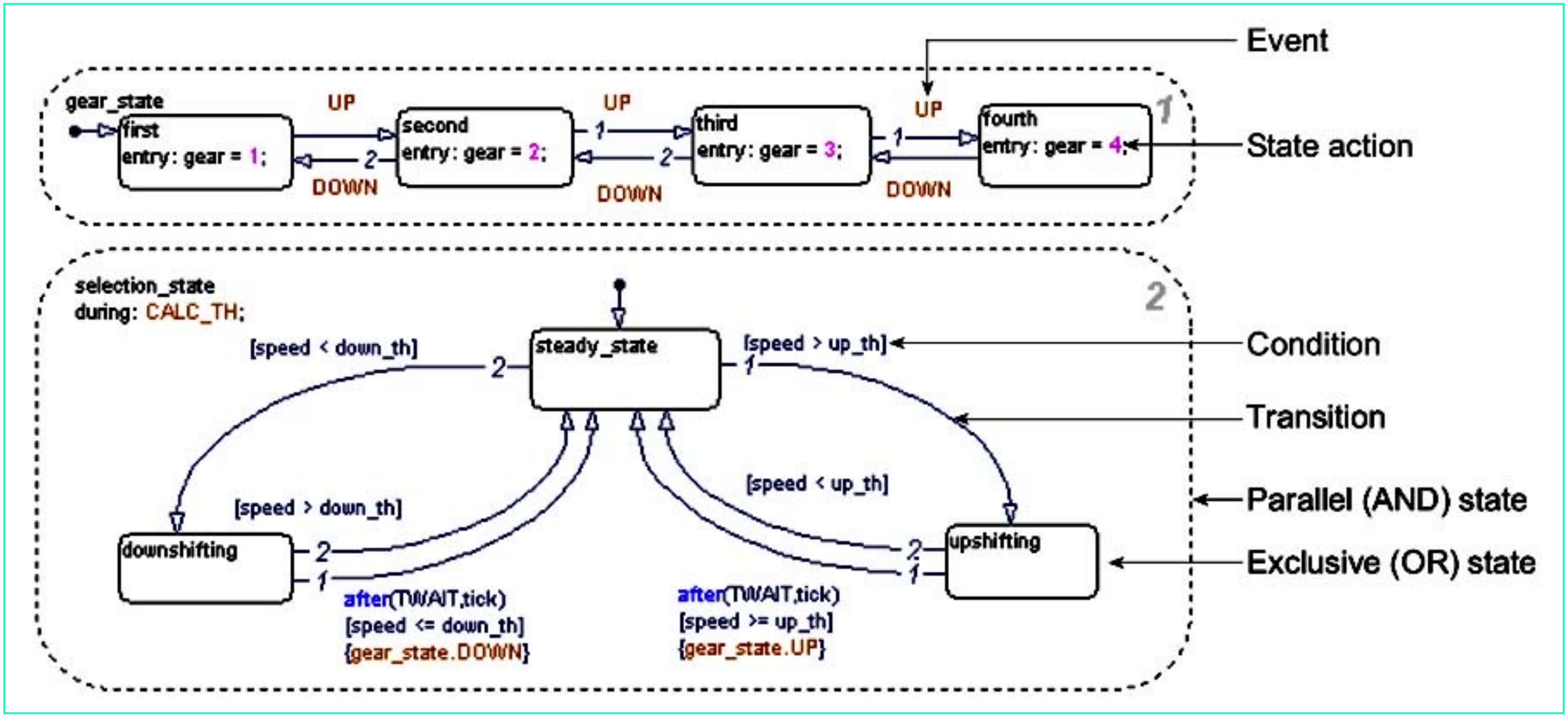
- Incorporation of event-driven logic in a control system
  - Simulink operates within the MATLAB environment
  - Stateflow implements logic blocks within Simulink

Automatic Shifting Example

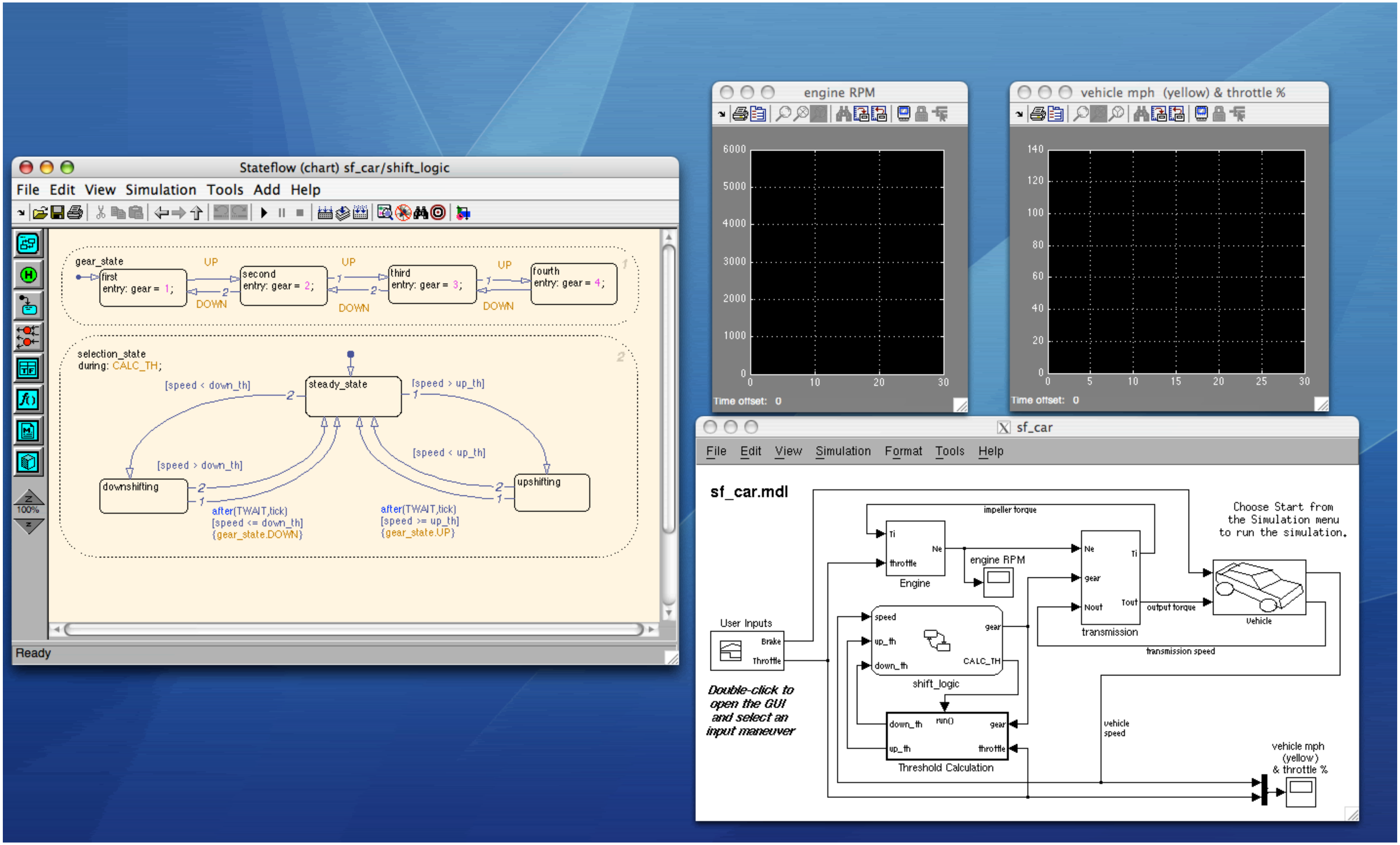
- Stateflow block represents the control logic
- Double-click on block to reveal the Stateflow logic



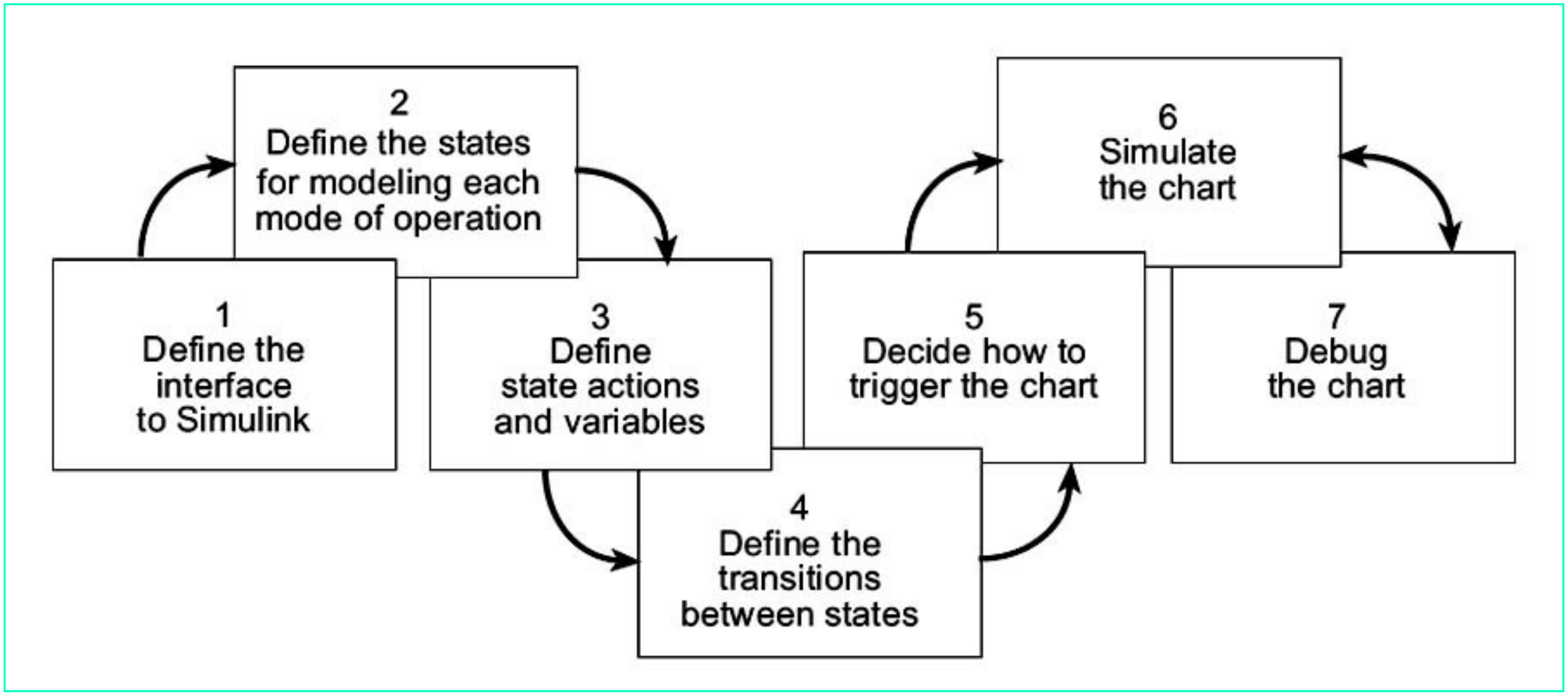
Stateflow Chart for an Automatic Transmission



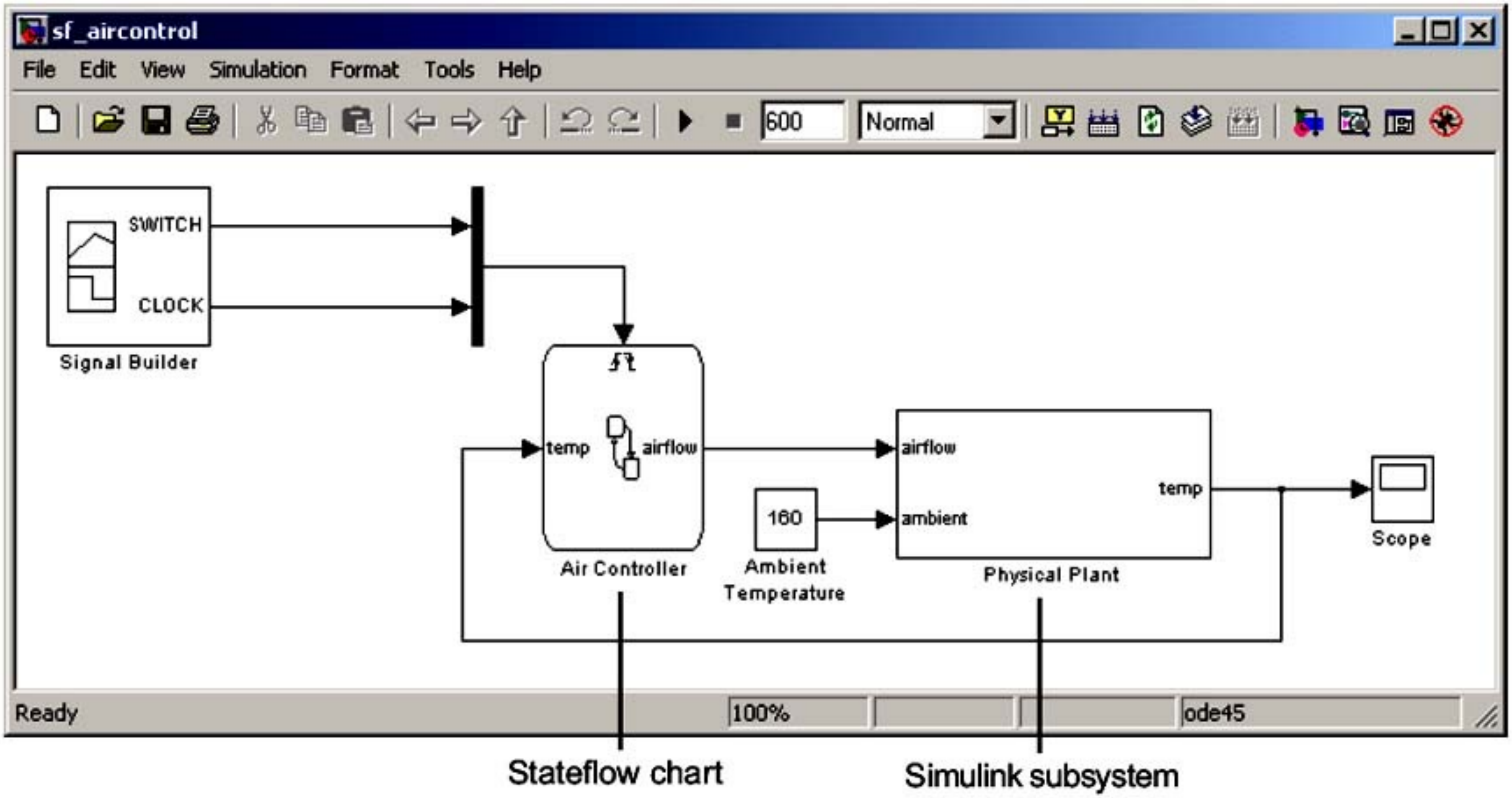
Automatic Shifting Simulation



Combining Discrete-Event Logic with the Dynamic Model



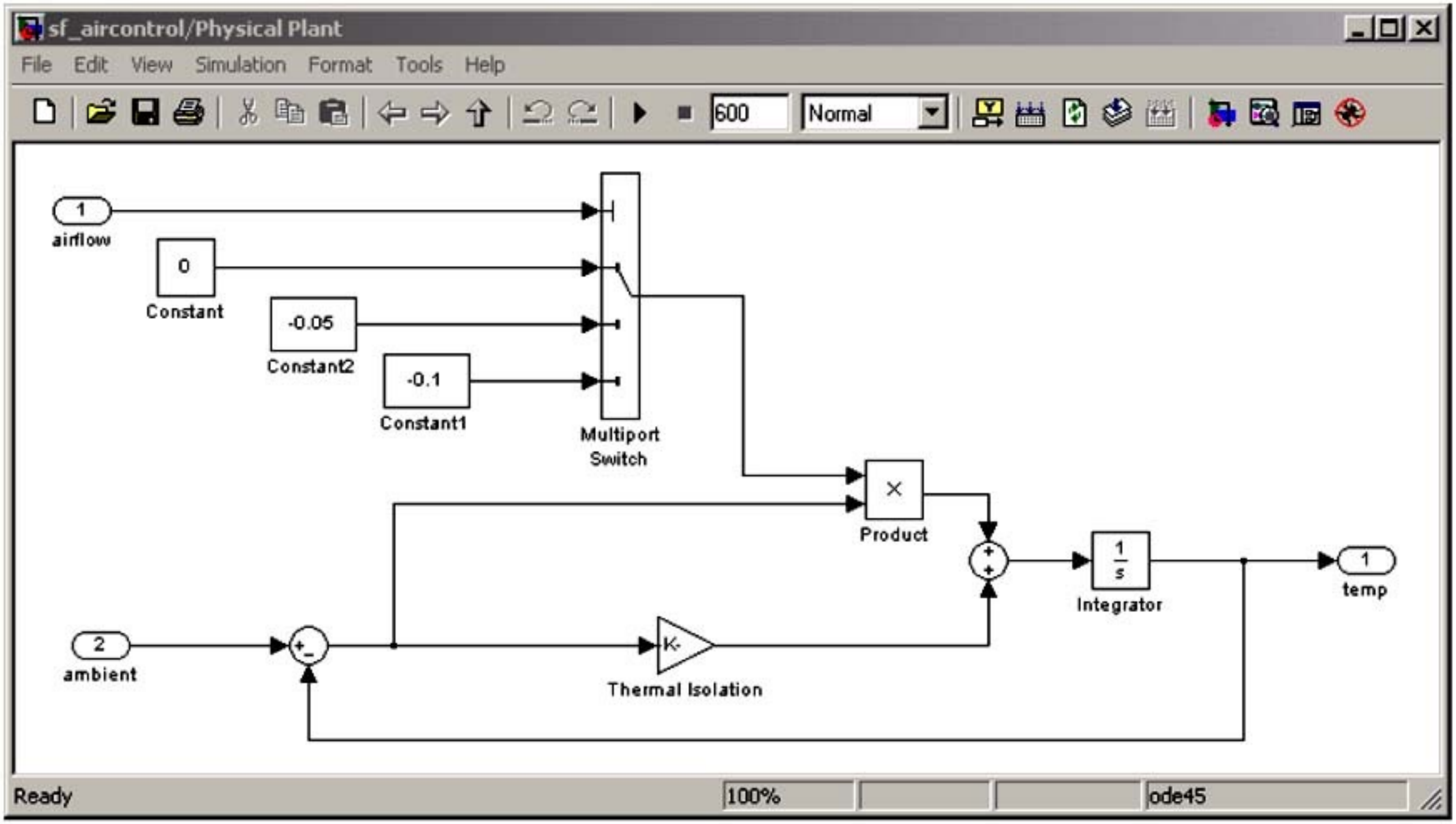
Temperature Control Example



See MATLAB Manual, *Getting Started*, Simulink, for details of model building
(http://www.mathworks.com/access/helpdesk/help/toolbox/stateflow/)

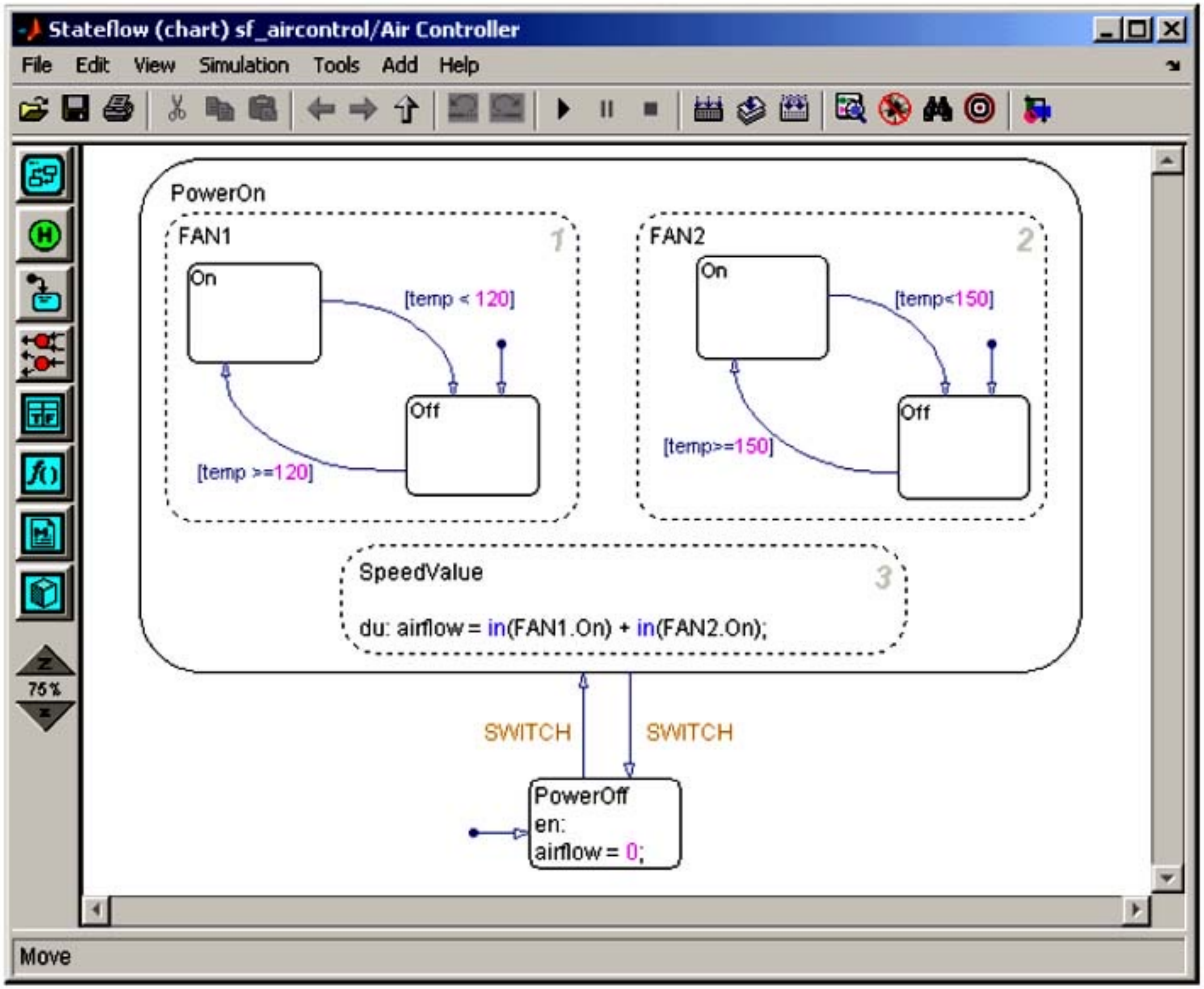
Physical Plant Model

Contents of *Physical Plant*

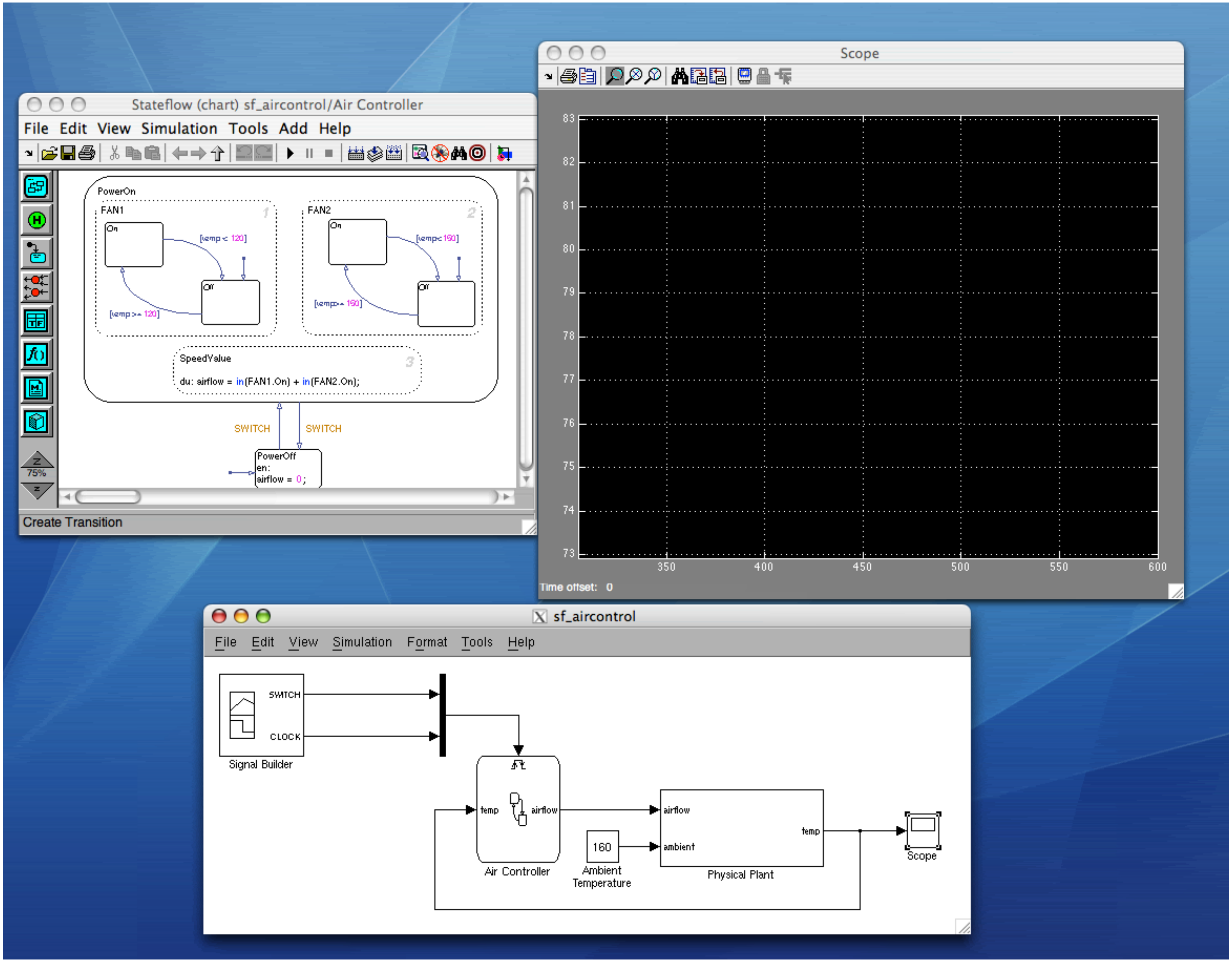


Air Control Logic

Contents of *Air Controller*



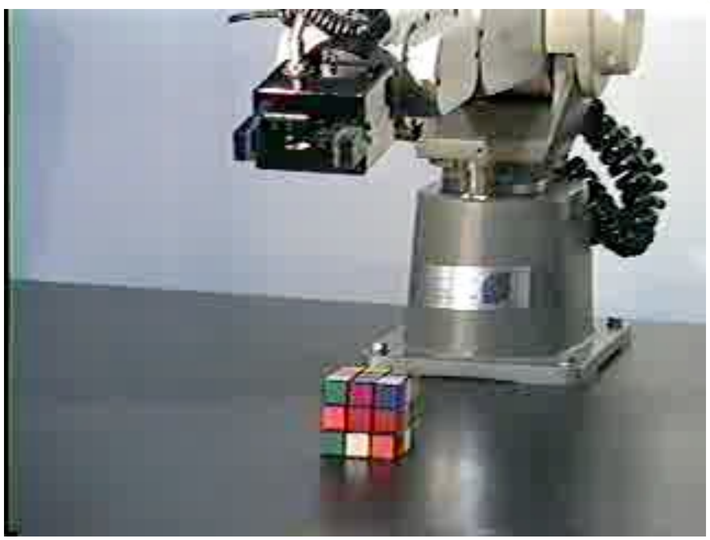
Temperature Control Simulation



Solving Rubik's Cube:

An algorithm

http://www.cs.swarthmore.edu/~knerr/helps/rcube.html



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