Propagation of Uncertainty in Dynamic Systems

Robert Stengel
Optimal Control and Estimation MAE 546
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- Propagation of the mean and variance in linear, time-varying discrete-time and continuous-time systems
- Markov processes and the transition function property
- Sampled-data representation of continuous-time systems
- White and colored noise inputs

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http://www.princeton.edu/~stengel/MAE546.html
http://www.princeton.edu/~stengel/OptConEst.html

Waves

- Deterministic and random components
- After a few seconds, the underlying model can be perceived
- But specifics of future waves cannot be predicted without random error

Linear, Time-Varying (LTV) Dynamic Model

- Discrete-time LTV model with known coefficients

$$\boxed{\mathbf{x}_k = \mathbf{\Phi}_{k-1}\mathbf{x}_{k-1} + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1} + \mathbf{\Lambda}_{k-1}\mathbf{w}_{k-1}}$$

- Random initial condition and disturbance inputs
- All random variables assumed to be Gaussian, i.e., they are fully described by means and covariances

Statistics of Random Variables

Gaussian uncertainty model for state mean and covariance

$$\boxed{\begin{aligned} \bar{\mathbf{x}}_0 &= E[\mathbf{x}_0]; \quad \mathbf{P}_0 = E\left\{[\mathbf{x}_0 - \bar{\mathbf{x}}_0][\mathbf{x}_0 - \bar{\mathbf{x}}_0]^T\right\} \\ \bar{\mathbf{x}}_k &= E[\mathbf{x}_k]; \quad \mathbf{P}_k = E\left\{[\mathbf{x}_k - \bar{\mathbf{x}}_k][\mathbf{x}_k - \bar{\mathbf{x}}_k]^T\right\} \end{aligned}}$$

Disturbance is assumed to have zero mean

$$\boxed{\bar{\mathbf{w}}_k = \mathbf{0}; \quad \mathbf{Q}_k = E\left\{[\mathbf{w}_k][\mathbf{w}_k]^T\right\}}$$

Control is assumed to be known without error

$$\boxed{\mathbf{u}_k = \bar{\mathbf{u}}_k; \quad E\left[\mathbf{u}_k - \bar{\mathbf{u}}_k\right] \triangleq \mathbf{U}_k = \mathbf{0}}$$

Multivariate Expected Values:

State

Mean value vector of state vector

$$\bar{\mathbf{x}} = E(\mathbf{x}) = \int_{-\infty}^{\infty} \mathbf{x}\, \mathrm{pr}(\mathbf{x})\, d\mathbf{x} = \begin{bmatrix} \bar{\mathbf{x}}_1 \\ \bar{\mathbf{x}}_2 \\ \cdots \\ \bar{\mathbf{x}}_n \end{bmatrix}$$

Covariance matrix of state vector

$$\mathbf{P} \triangleq E\left[(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^T\right] = \int_{-\infty}^{\infty} (\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^T\, \mathrm{pr}(\mathbf{x})\, d\mathbf{x}$$

Probability density function of the state

$$\mathrm{pr}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\,|\mathbf{P}|^{1/2}}\, e^{-\frac{1}{2}(\mathbf{x}-\bar{\mathbf{x}})^T \mathbf{P}^{-1}(\mathbf{x}-\bar{\mathbf{x}})}$$

Covariance Matrix of State Vector

$$\mathbf{P} = E\left[(\mathbf{x}-\bar{\mathbf{x}})(\mathbf{x}-\bar{\mathbf{x}})^T\right] \triangleq \begin{bmatrix} \sigma_{x_1}{}^2 & \cdots & \rho_{1n}\sigma_{x_1}\sigma_{x_n} \\ \cdots & \cdots & \cdots \\ \rho_{1n}\sigma_{x_1}\sigma_{x_n} & \cdots & \sigma_{x_n}{}^2 \end{bmatrix}$$

$\sigma_{x_1}{}^2$ = Variance of x_1

ρ_{12} = Correlation coefficient for x_1 and x_2, in $(-1,+1)$

$\rho_{12}\sigma_{x_1}\sigma_{x_2}$ = Covariance of x_1 and x_2

Gaussian probability distribution is completely described by its mean value and covariance matrix

Initial Condition and Control of the LTV Dynamic Model

Initial condition described by mean and covariance

$$E(\mathbf{x}_0) = \bar{\mathbf{x}}_0 \triangleq \mathbf{m}_o \; ; \quad E\left[(\mathbf{x}_0 - \mathbf{m}_o)(\mathbf{x}_0 - \mathbf{m}_o)^T\right] \triangleq \mathbf{P}_0$$

$$\dim(\mathbf{x}_o) = n \times 1$$

Control input is known precisely

$$E[\mathbf{u}_k] = \bar{\mathbf{u}}_k = \mathbf{u}_k; \quad E\left[(\mathbf{u}_k - \bar{\mathbf{u}}_k)(\mathbf{u}_k - \bar{\mathbf{u}}_k)^T\right] = \mathbf{U}_k = \mathbf{0}$$

$$\dim(\mathbf{u}_k) = m \times 1$$

Cross-covariances are zero

$$E\left[(\mathbf{x}_k - \bar{\mathbf{x}}_k)\mathbf{w}_k^T\right] = \mathbf{M}_k = \mathbf{0}$$

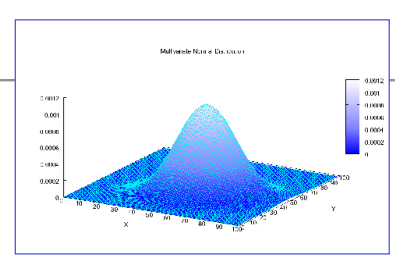
$$E\left[(\mathbf{x}_k - \bar{\mathbf{x}}_k)\mathbf{u}_k^T\right] = \mathbf{0}$$

$$E\left[\mathbf{w}_k \mathbf{u}_k^T\right] = \mathbf{0}$$

Probability Density Function of the LTV Dynamic Model

Initial probability density function depends only on the mean and covariance of the Gaussian distribution

$$\mathrm{pr}(\mathbf{x}_0) = \frac{1}{(2\pi)^{n/2}\left|\mathbf{P}_0\right|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}_0 - \mathbf{m}_0)^T \mathbf{P}_0^{-1}(\mathbf{x}_0 - \mathbf{m}_0)}$$



Multivariate Expected Values:
Disturbance

Mean value vector of disturbance vector

$$\bar{\mathbf{w}} = E(\mathbf{w}) = \int_{-\infty}^{\infty} \mathbf{w}\ \mathrm{pr}(\mathbf{w})\, d\mathbf{w} = \begin{bmatrix} \bar{\mathbf{w}}_1 \\ \bar{\mathbf{w}}_2 \\ \cdots \\ \bar{\mathbf{w}}_s \end{bmatrix}$$

Covariance matrix of disturbance vector

$$\mathbf{Q} \triangleq E\left[(\mathbf{w}-\bar{\mathbf{w}})(\mathbf{w}-\bar{\mathbf{w}})^T\right] = \int_{-\infty}^{\infty} (\mathbf{w}-\bar{\mathbf{w}})(\mathbf{w}-\bar{\mathbf{w}})^T\ \mathrm{pr}(\mathbf{w})\, d\mathbf{w}$$

Probability density function of the disturbance

$$\mathrm{pr}(\mathbf{w}) = \frac{1}{(2\pi)^{s/2}\,|\mathbf{Q}|^{1/2}}\, e^{-\frac{1}{2}(\mathbf{w}-\bar{\mathbf{w}})\mathbf{Q}^{-1}(\mathbf{w}-\bar{\mathbf{w}})}$$

Covariance Matrix of Disturbance Vector

$$\mathbf{Q} = E\left[(\mathbf{w}-\bar{\mathbf{w}})(\mathbf{w}-\bar{\mathbf{w}})^T\right] \triangleq \begin{bmatrix} \sigma_{w_1}{}^2 & \cdots & \rho_{1s}\sigma_{w_1}\sigma_{w_s} \\ \cdots & \cdots & \cdots \\ \rho_{1s}\sigma_{w_1}\sigma_{w_r} & \cdots & \sigma_{w_s}{}^2 \end{bmatrix}$$

$\sigma_{w_1}{}^2$ = Variance of w_1

ρ_{12} = Correlation coefficient for w_1 and w_2, in $(-1,+1)$

$\rho_{12}\sigma_{w_1}\sigma_{w_2}$ = Covariance of w_1 and w_2

Disturbance Input of the LTV Dynamic Model

Disturbance input ('process noise') is a white-noise sequence

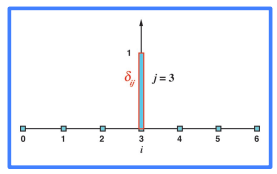
$$E\left(\mathbf{w}_k\right) = \mathbf{0}$$

$$E\left(\mathbf{w}_k \mathbf{w}_k^T\right) = \mathbf{Q}'_k; \quad E\left(\mathbf{w}_k \mathbf{w}_{k-l}^T\right) = \mathbf{0}, \quad l = \text{ any non-zero integer}$$

or

$$E\left(\mathbf{w}_j \mathbf{w}_k^T\right) = \mathbf{Q}'_k \delta_{jk}$$

$$\delta_{jk} \triangleq \textbf{Kronecker delta function} = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$



Propagation of the Ensemble Mean Value Estimate

Expected value of the state

$$E\left(\mathbf{x}_{k+1}\right) = E\left(\mathbf{\Phi}\mathbf{x}_k + \mathbf{\Gamma}\mathbf{u}_k + \mathbf{\Lambda}\mathbf{w}_k\right)$$

$$\bar{\mathbf{x}}_{k+1} = \mathbf{\Phi}\bar{\mathbf{x}}_k + \mathbf{\Gamma}\bar{\mathbf{u}}_k + 0, \quad \bar{\mathbf{x}}_0 \text{ given}$$

Propagation of the Ensemble State Covariance Estimate

State covariance matrix

Expected values of cross terms are zero

$$\mathbf{P}_{k+1} = E\left[\left(\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1}\right)\left(\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1}\right)^T\right]$$

$$\mathbf{P}_{k+1} = E\left\{\left[\mathbf{\Phi}\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right) + \mathbf{\Gamma}\left(\mathbf{u}_k - \bar{\mathbf{u}}_k\right) + \mathbf{\Lambda}\mathbf{w}_k\right]\left[\mathbf{\Phi}\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right) + \mathbf{\Gamma}\left(\mathbf{u}_k - \bar{\mathbf{u}}_k\right) + \mathbf{\Lambda}\mathbf{w}_k\right]^T\right\}$$

$$\begin{aligned}
\mathbf{P}_{k+1} &= E\left[\mathbf{\Phi}\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right)\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right)^T{}_k\,\mathbf{\Phi}^T + \mathbf{0} + \mathbf{\Lambda}\mathbf{w}_k\mathbf{w}^T{}_k\mathbf{\Lambda}^T{}_k\right] \\
&= \mathbf{\Phi}E\left[\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right)\left(\mathbf{x}_k - \bar{\mathbf{x}}_k\right)^T{}_k\right]\mathbf{\Phi}^T + \mathbf{\Lambda}E\left(\mathbf{w}_k\mathbf{w}^T{}_k\right)\mathbf{\Lambda}^T \\
&= \mathbf{\Phi}\mathbf{P}_k\mathbf{\Phi}^T + \mathbf{\Lambda}\mathbf{Q}_\mathbf{k}\mathbf{\Lambda}^T, \quad \mathbf{P}_0 \text{ given}
\end{aligned}$$

Propagation of the Estimated State Probability Density Function

Uncertainty propagation model is a <u>Markov Process</u>

$$\bar{\mathbf{x}}_{k+1} = \mathbf{\Phi}\bar{\mathbf{x}}_k + \mathbf{\Gamma}\bar{\mathbf{u}}_k, \quad \bar{\mathbf{x}}_0 \text{ given}$$

$$\mathbf{P}_{k+1} = \mathbf{\Phi}\mathbf{P}_k\mathbf{\Phi}^T + \mathbf{\Lambda}\mathbf{Q}_\mathbf{k}\mathbf{\Lambda}^T, \quad \mathbf{P}_0 \text{ given}$$

Propagating the state mean and covariance is equivalent to propagating the entire probability density function of the state

$$\mathrm{pr}(\mathbf{x}_{k+1}) = \frac{1}{\left(2\pi\right)^{n/2}\left|\mathbf{P}_{k+1}\right|^{1/2}} e^{-\frac{1}{2}\left(\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1}\right)^T \mathbf{P}_{k+1}{}^{-1}\left(\mathbf{x}_{k+1} - \bar{\mathbf{x}}_{k+1}\right)}$$

Expected Value of the State

First moment of x

$$E(\mathbf{x}_k) = E\left(\mathbf{\Phi}_{k-1}\mathbf{x}_{k-1} + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1} + \mathbf{\Lambda}_{k-1}\mathbf{w}_{k-1}\right)$$

$$\bar{\mathbf{x}}_k = \mathbf{\Phi}_{k-1}\bar{\mathbf{x}}_{k-1} + \mathbf{\Gamma}_{k-1}\bar{\mathbf{u}}_{k-1} + \mathbf{\Lambda}_{k-1}(0)$$

$$\mathbf{m}_k = \mathbf{\Phi}_{k-1}\mathbf{m}_{k-1} + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1}$$

Expected Value of the State Covariance

Second central moment of x

$$\mathbf{P}_k \triangleq E\left[(\mathbf{x}_k - \mathbf{m}_k)(\mathbf{x}_k - \mathbf{m}_k)^T\right]$$

$$= E\left\{\left[\mathbf{\Phi}_{k-1}(\mathbf{x}_{k-1} - \mathbf{m}_{k-1}) + \mathbf{\Gamma}_{k-1}(\mathbf{u}_{k-1} - \bar{\mathbf{u}}_{k-1}) + \mathbf{\Lambda}_{k-1}\mathbf{w}_{k-1}\right]\left[\mathbf{\Phi}_{k-1}(\mathbf{x}_{k-1} - \mathbf{m}_{k-1}) + \mathbf{\Gamma}_{k-1}(\mathbf{u}_{k-1} - \bar{\mathbf{u}}_{k-1}) + \mathbf{\Lambda}_{k-1}\mathbf{w}_{k-1}\right]^T\right\}$$

With negligible cross-covariance

$$\mathbf{P}_k = \mathbf{\Phi}_{k-1}E\left[(\mathbf{x}_{k-1} - \mathbf{m}_{k-1})(\mathbf{x}_{k-1} - \mathbf{m}_{k-1})^T\right]\mathbf{\Phi}_{k-1}{}^T + \mathbf{\Lambda}_{k-1}E\left(\mathbf{w}_{k-1}\mathbf{w}^T{}_{k-1}\right)\mathbf{\Lambda}_{k-1}{}^T$$

$$= \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + \mathbf{\Lambda}_{k-1}\mathbf{Q}'_{k-1}\mathbf{\Lambda}_{k-1}^T$$

$$\triangleq \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + \mathbf{Q}_{k-1}$$

$$\mathbf{Q}_{k-1} \triangleq \mathbf{\Lambda}_{k-1}\mathbf{Q}'_{k-1}\mathbf{\Lambda}_{k-1}^T$$

$$\dim(\mathbf{Q}_{k-1}) = n \times n$$

$$\dim(\mathbf{Q}'_{k-1}) = s \times s$$

Probability Mass and Density Functions

The random sequence

- has the <u>transition function property,</u> i.e., current value conditioned on prior value

- is a Gauss-Markov sequence

Probability Mass Function

$$\Pr(\Delta\mathbf{x}_k) = \Pr(\Delta\mathbf{x}_k \mid \Delta\mathbf{x}_{k-1})\,\Pr(\Delta\mathbf{x}_{k-1})$$

$$= \Pr(\Delta\mathbf{x}_k \mid \Delta\mathbf{x}_{k-1})\,\Pr(\Delta\mathbf{x}_{k-1} \mid \Delta\mathbf{x}_{k-2})\,\Pr(\Delta\mathbf{x}_{k-2}) = \left[\prod_{i=1}^{k} \Pr(\Delta\mathbf{x}_i \mid \Delta\mathbf{x}_{i-1})\right]\Pr(\Delta\mathbf{x}_0)$$

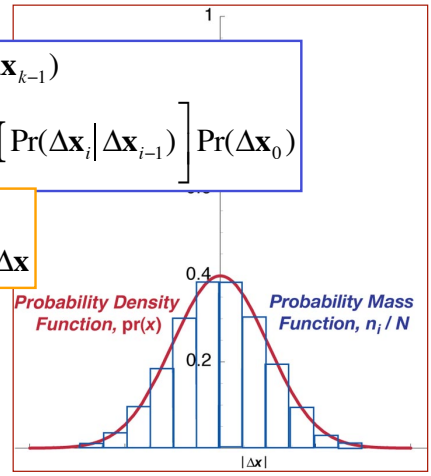
$$\Pr(\Delta\mathbf{x}_k) \approx \mathrm{pr}(\mathbf{x}_k)\Delta\mathbf{x}$$

$$\Pr(\Delta\mathbf{x}_k \mid \Delta\mathbf{x}_{k-1})\Pr(\Delta\mathbf{x}_{k-1}) \approx \mathrm{pr}(\Delta\mathbf{x}_k \mid \Delta\mathbf{x}_{k-1})\mathrm{pr}(\mathbf{x}_{k-1})\Delta\mathbf{x}$$

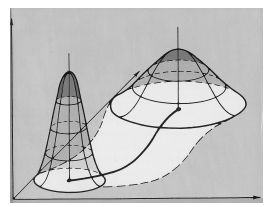
Probability Density Function

$$\lim_{\Delta\mathbf{x}\to 0} \Pr(\Delta\mathbf{x}_k) = \lim_{\Delta\mathbf{x}\to 0} \Pr(\Delta\mathbf{x}_k \mid \Delta\mathbf{x}_{k-1})\,\Pr(\Delta\mathbf{x}_{k-1})$$

$$= \mathrm{pr}(\mathbf{x}_k) = \mathrm{pr}(\mathbf{x}_k \mid \mathbf{x}_{k-1})\,\mathrm{pr}(\mathbf{x}_{k-1})$$



Conditional Probability Density Function of the State



$$\mathrm{pr}(\mathbf{x}_k) = \frac{1}{(2\pi)^{n/2}\,|\mathbf{P}_k|^{1/2}}\, e^{-\frac{1}{2}(\mathbf{x}_k-\mathbf{m}_k)^T \mathbf{P}_k^{-1}(\mathbf{x}_k-\mathbf{m}_k)}$$

$$\mathbf{m}_k = \mathbf{\Phi}_{k-1}\mathbf{m}_{k-1} + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1}$$

$$\mathbf{P}_k = \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}^{\ T} + \mathbf{Q}_{k-1}$$

The density function is conditioned on the prior state

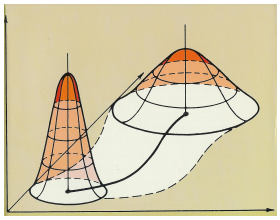
$$\mathrm{pr}(\mathbf{x}_k \mid \mathbf{x}_{k-1}) =$$

$$\frac{1}{(2\pi)^{n/2}\left|\mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}^{\ T} + \mathbf{Q}_{k-1}\right|^{1/2}}\, e^{-\frac{1}{2}\left[\mathbf{x}_k-(\mathbf{\Phi}_{k-1}\mathbf{m}_{k-1}+\mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1})\right]^T\left(\mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}^{\ T}+\mathbf{Q}_{k-1}\right)^{-1}\left[\mathbf{x}_k-(\mathbf{\Phi}_{k-1}\mathbf{m}_{k-1}+\mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1})\right]}$$

... and propagation is a Markov process

$$\mathrm{pr}(\mathbf{x}_k) = \mathrm{pr}(\mathbf{x}_k \mid \mathbf{x}_{k-1})\mathrm{pr}(\mathbf{x}_{k-1}) = \mathrm{pr}(\mathbf{x}_k \mid \mathbf{x}_{k-1}) \text{ if } \mathrm{pr}(\mathbf{x}_{k-1}) = 1$$

Gauss-Markov Sequence



$$\mathbf{m}_k = \mathbf{\Phi}_{k-1}\mathbf{m}_{k-1} + \mathbf{\Gamma}_{k-1}\mathbf{u}_{k-1}$$
$$\mathbf{P}_k = \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}^{\ T} + \mathbf{Q}_{k-1}$$

- **The mean and covariance are completely specified by the prior probability distribution**
- **The random sequence**
  - **has the <u>transition function</u> property, i.e., current value *(pdf)* conditioned on prior *pdf***
  - **is a Gauss-Markov sequence**

$$\mathrm{pr}(\mathbf{x}_k) = \mathrm{pr}(\mathbf{x}_k|\mathbf{x}_{k-1})\,\mathrm{pr}(\mathbf{x}_{k-1}) = \mathrm{pr}(\mathbf{x}_k|\mathbf{x}_{k-1})\,\mathrm{pr}(\mathbf{x}_{k-1}|\mathbf{x}_{k-2})\,\mathrm{pr}(\mathbf{x}_{k-2}) = \cdots$$
$$= \left[\prod_{i=1}^{k}\mathrm{pr}(\mathbf{x}_i|\mathbf{x}_{i-1})\right]\mathrm{pr}(\mathbf{x}_0)$$

Sampled-Data Representation of Continuous-Time Systems

Sampled-Data Representation of Continuous-Time Systems

Continuous-time LTV model with known coefficients

$$\dot{\mathbf{x}}(t) = \mathbf{F}(t)\mathbf{x}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{L}(t)\mathbf{w}(t), \quad \mathbf{x}(t_o)\ given$$

$$\mathbf{x}(t) = \mathbf{x}(t_o) + \int_{t_o}^{t} \left[\mathbf{F}(\tau)\mathbf{x}(\tau) + \mathbf{G}(\tau)\mathbf{u}(\tau) + \mathbf{L}(\tau)\mathbf{w}(\tau)\right] d\tau$$

Incremental solution

$$\mathbf{x}(t_k) = \mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \left[\mathbf{F}\mathbf{x}(\tau) + \mathbf{G}\mathbf{u}(\tau) + \mathbf{L}\mathbf{w}(\tau)\right] d\tau$$

$$= \mathbf{\Phi}\left(t_k, t_{k-1}\right)\mathbf{x}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \mathbf{\Phi}\left(t_k, \tau\right)\left[\mathbf{G}\left(\tau\right)\mathbf{u}(\tau) + \mathbf{L}\left(\tau\right)\mathbf{w}(\tau)\right] d\tau$$

Descriptions of Random Variables

$$E\left(\mathbf{x}_0\right) = \mathbf{m}_o\ ; \quad E\left[\left(\mathbf{x}_0 - \mathbf{m}_o\right)\left(\mathbf{x}_0 - \mathbf{m}_o\right)^T\right] = \mathbf{P}_0$$

$$E\left[\mathbf{w}\left(t\right)\right] = \mathbf{0}$$

$$E\left[\mathbf{w}\left(t\right)\mathbf{w}^T\left(\tau\right)\right] = \mathbf{Q}'\delta\left(t - \tau\right)$$

$$E\left[\mathbf{u}\left(t\right)\right] = \mathbf{u}\left(t\right); \quad E\left\{\left[\mathbf{u}\left(t\right) - \bar{\mathbf{u}}\left(t\right)\right]\left[\mathbf{u}\left(t\right) - \bar{\mathbf{u}}\left(t\right)\right]^T\right\} = \mathbf{0}$$

Mean Value and Covariance Solutions

Mean value propagation from t_{k-1} to t_k

$$\mathbf{m}_k = \mathbf{\Phi}_k \mathbf{m}_{k-1} + \int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{G}(\tau)\mathbf{u}(\tau)d\tau$$

Covariance propagation

Double integration over time $(\tau \text{ and } \alpha)$

$$\mathbf{P}_k = \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + E\left\{\left[\int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{L}(\tau)\mathbf{w}(\tau)d\tau\right]\left[\int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\alpha)\mathbf{L}(\alpha)\mathbf{w}(\alpha)d\alpha\right]^T\right\}$$

$$= \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{L}(\tau)E\left[\mathbf{w}(\tau)\mathbf{w}^T(\alpha)\right]\mathbf{L}^T(\alpha)\mathbf{\Phi}^T(t_k,\alpha)d\tau\, d\alpha$$

Covariance Propagation

$$E\left[\mathbf{w}(\tau)\mathbf{w}^T(\alpha)\right] \triangleq \mathbf{Q}'_C\,\delta(\tau-\alpha)$$

$$\mathbf{P}_k = \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{L}(\tau)\mathbf{Q}'_C\,\delta(\tau-\alpha)\mathbf{L}^T(\alpha)\mathbf{\Phi}^T(t_k,\alpha)d\tau\, d\alpha$$

$$= \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + \int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{L}(\tau)\mathbf{Q}'_C\,\mathbf{L}^T(\tau)\mathbf{\Phi}^T(t_k,\tau)d\tau$$

$$\triangleq \mathbf{\Phi}_{k-1}\mathbf{P}_{k-1}\mathbf{\Phi}_{k-1}{}^T + \mathbf{Q}_{k-1}$$

Relationship Between Discrete- and Continuous-Time Disturbance Models

$$\mathbf{Q}_{k-1} = \int_{t_{k-1}}^{t_k} \mathbf{\Phi}(t_k,\tau)\mathbf{L}(\tau)\mathbf{Q}'_C\,\mathbf{L}^T(\tau)\mathbf{\Phi}^T(t_k,\tau)\,d\tau$$

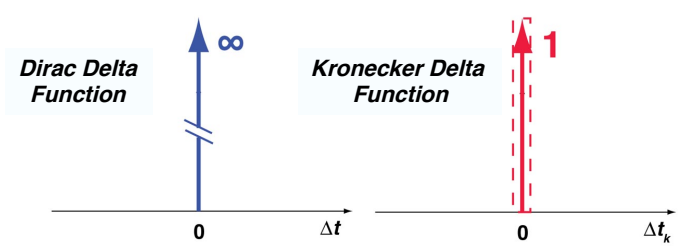
$\mathbf{Q}_{k-1}$: **Covariance matrix** $(n \times n)$

$\mathbf{Q}'_C$: **Spectral density matrix** $(s \times s)$

For small Δt

$$\begin{aligned}\mathbf{Q}_{k-1} &\approx \mathbf{L}(t_{k-1})\mathbf{Q}'_C(t_{k-1})\mathbf{L}^T(t_{k-1})\,\Delta t \\ &\neq \mathbf{L}(t_{k-1})\mathbf{Q}'_C(t_{k-1})\mathbf{L}^T(t_{k-1})\,\Delta t^2\end{aligned}$$

Autocovariance Functions for Continuous-and Discrete-Time White Noise *(Scalar)*



$$\phi_{ww}(\Delta t) = \mathbf{Q}'_C\,\delta(\Delta t)$$

$$\phi_{ww}(\Delta t_k) = \mathbf{Q}_{k-1}\delta(\Delta t_k)$$

$$\delta(\Delta t) = \begin{cases} \infty, & \Delta t = 0 \\ 0, & \Delta t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \lim_{\Delta t_1 \to 0} \int_{-\Delta t_1}^{\Delta t_1} \delta(\Delta t)\,d(\Delta t) = 1$$

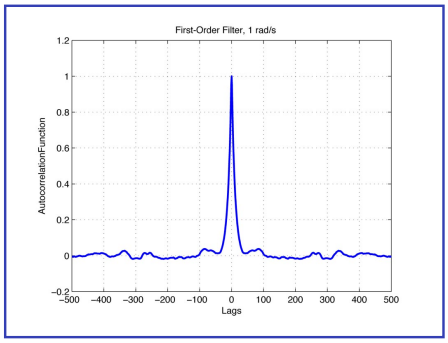
$$\delta_{jk} = \delta(\Delta t_k) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}$$

Colored Noise, Variance, Spectral Density Matrices *(Scalar)*

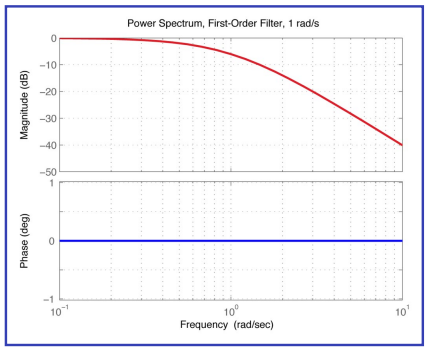
- **Spreading of the autocovariance function is accompanied by**
  - **Finite variance** $\phi_{xx}(\tau)$
  - **Low-pass filtering of the power spectral density**

$$\Phi_{xx}(\omega) = \int_{-\infty}^{\infty} \phi_{xx}(\tau) e^{-j\omega\tau} \, d\tau$$

Autocovariance Function



Power Spectral Density Function



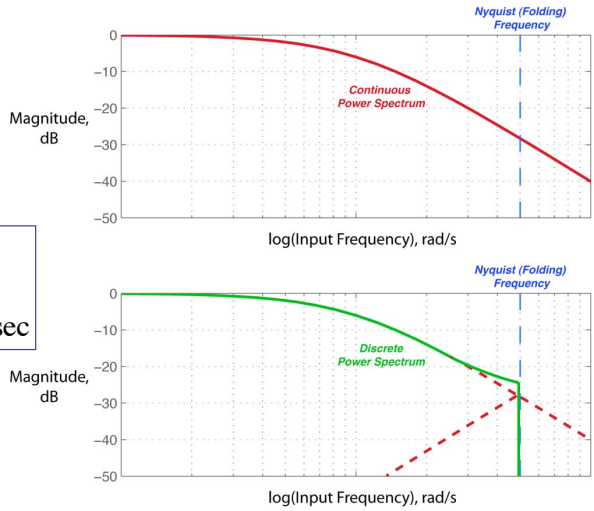
Spectral Density Matrices

- **Dynamic systems subject all white-noise inputs to low-pass filtering**
- **Nyquist (or folding) frequency**

$$\omega_{Nyquist} = \frac{1}{2}\omega_{Sampling} = \frac{\pi}{T}$$

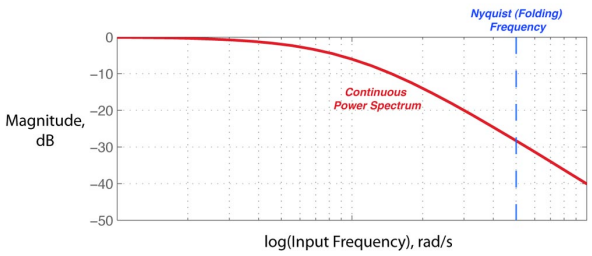
$$T = \text{Sampling interval, sec}$$

- **Signals above this frequency fold and corrupt lower-frequency sampled signals (aliasing)**

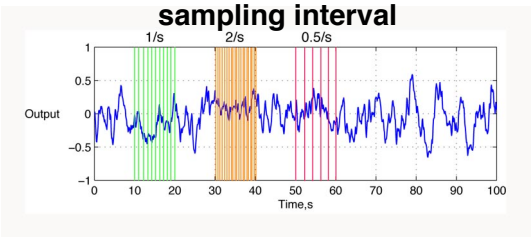


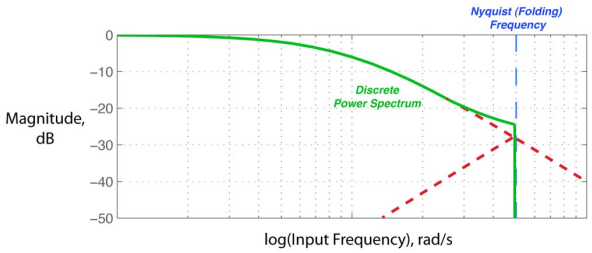
<u>Caveat</u>: Covariance and Spectra capture "stochastic equilibrium" effects but not transient effects

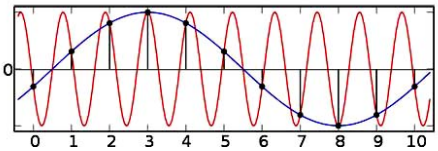
Frequency Folding (Aliasing)



Folding frequency shifts with sampling interval

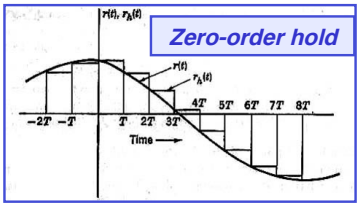


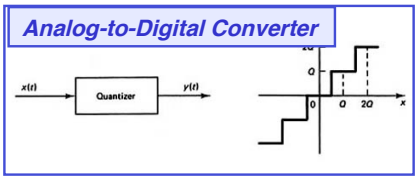


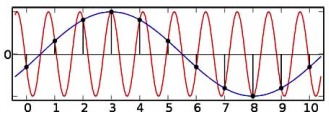


Quantization, Zero-Order Hold, and Inter-Sample Ripple

- Digital control subject to zero-order hold in D/A
- Sampled signal misses inter-sample transient response
- *Effective delay* of sampled signal
- Continuous signal sampled with finite precision in A/D
- Inter-sample ripple







Colored Noise Disturbance

- Disturbance may not be "white" noise
  - Power may vary with frequency
- Mean

$$E(\mathbf{w}_k) = \mathbf{0}$$

$$\dim(\mathbf{w}_k) = s \times 1$$

- Covariance

$$E(\mathbf{w}_k \mathbf{w}_k^T) = \mathbf{W}_k$$

- Correlation with adjacent signal

$$E(\mathbf{w}_k \mathbf{w}_{k-1}^T) \neq \mathbf{0} = \mathbf{V}_k$$

- Linear model for disturbance propagation
  - Driven by white noise sequence

$$\mathbf{w}_k = \mathbf{A}_{k-1}\mathbf{w}_{k-1} + \boldsymbol{\eta}_{k-1}$$

$$E(\boldsymbol{\eta}_k) = \mathbf{0}$$

$$E(\boldsymbol{\eta}_j \boldsymbol{\eta}_k^T) = \mathbf{Q}_{\boldsymbol{\eta}_k} \delta_{jk}$$

Calculation of Colored Disturbance Process Parameters

Autocovariance of scalar Markov process

$$\phi_{xx}(1) = E(x_i x_{i+1}) = bE(x_i^2) = b\sigma_x^2$$

$$\phi_{xx}(k) = b^{|k|}\sigma_x^2 = b^{|k|}\sigma_u^2$$

Autocovariance of vector Markov process

$$\begin{aligned} E(\mathbf{w}_k \mathbf{w}_{k-1}^T) &= E\left[(\mathbf{A}_{k-1}\mathbf{w}_{k-1} + \boldsymbol{\eta}_{k-1})\mathbf{w}_{k-1}^T\right] \\ &= \mathbf{A}_{k-1} E(\mathbf{w}_{k-1}\mathbf{w}_{k-1}^T) \\ &= \mathbf{A}_{k-1}\mathbf{W}_{k-1} \\ &\triangleq \mathbf{V}_k \end{aligned}$$

Therefore, state transition matrix of the noise model is

$$\mathbf{A}_{k-1} = \mathbf{V}_k \mathbf{W}_{k-1}^{-1}$$

Propagating the Colored Disturbance

$$\mathbf{W}_k = \mathbf{A}_{k-1}\mathbf{W}_{k-1}\mathbf{A}_{k-1}^T + \mathbf{Q}_{\boldsymbol{\eta}_k}$$

- System state vector is augmented to include the disturbance

$$\mathbf{x}'_k \triangleq \begin{bmatrix} \mathbf{m}_k \\ \mathbf{w}_k \end{bmatrix}, \quad \dim(\mathbf{x}'_k) = (n+s)\times 1$$

- Augmented equation for the mean

$$\begin{bmatrix} \mathbf{m}_k \\ \mathbf{w}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{k-1} & \boldsymbol{\Lambda}_{k-1} \\ \mathbf{0} & \mathbf{A}_{k-1} \end{bmatrix} \begin{bmatrix} \mathbf{m}_{k-1} \\ \mathbf{w}_{k-1} \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Gamma}_{k-1} \\ \mathbf{0} \end{bmatrix} \mathbf{u}_{k-1} + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_s \end{bmatrix} \boldsymbol{\eta}_{k-1}$$

- Augmented covariance equation

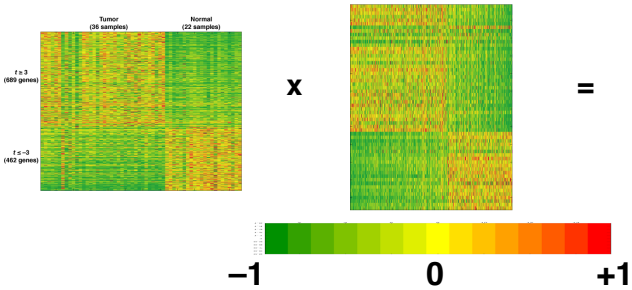
$$\begin{bmatrix} \mathbf{P}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_k \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}_{k-1} & \boldsymbol{\Lambda}_{k-1} \\ \mathbf{0} & \mathbf{A}_{k-1} \end{bmatrix} \begin{bmatrix} \mathbf{P}_{k-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_{k-1} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Phi}_{k-1} & \boldsymbol{\Lambda}_{k-1} \\ \mathbf{0} & \mathbf{A}_{k-1} \end{bmatrix}^T + \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_s \end{bmatrix} \mathbf{Q}_{\boldsymbol{\eta}_{k-1}} \begin{bmatrix} \mathbf{0} & \mathbf{I}_s \end{bmatrix}$$

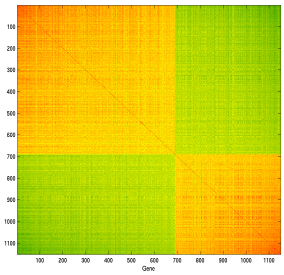
Next Time:
Kalman Filter for Discrete-Time Systems

Supplemental Material

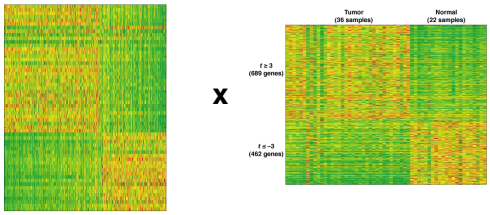
Covariance Matrix is Expected Value of the Outer Product

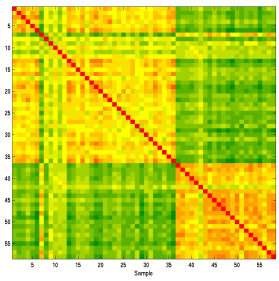
- **Transcript correlation for normal and abnormal samples**
  - **Identifies possible components of biological circuits**





- **Sample tissue correlation for RNA expression**
  - **Confirms or questions the classification of samples**





Example: Propagating a Scalar Probability Density Function

Scalar LTI system with zero-mean Gaussian random input

$$x_{k+1} = bx_k + \sqrt{1-b^2}\,u_k + \sqrt{1-b^2}\,w_k, \quad x_0 \ given$$

Propagation of the mean value

$$\bar{x}_{i+1} = b\bar{x}_i + \sqrt{1-b^2}\,\bar{u}_i, \quad \bar{x}_0 \ given$$

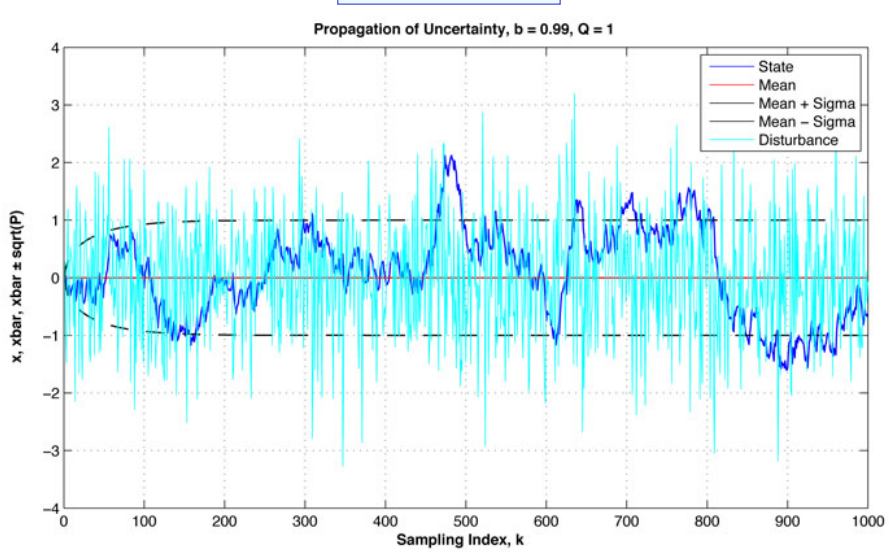
Propagation of the variance

$$P_{k+1} = b^2 P_k + \left(1-b^2\right)Q_k, \quad P_0 \ given$$

$$Q_k = E\left(w_k^2\right)$$

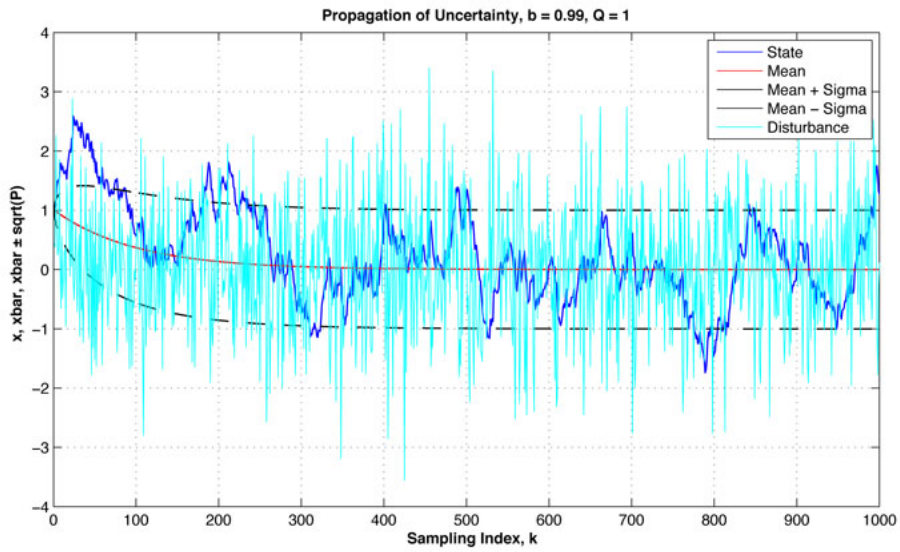
Example: Propagating a Scalar Probability Density Function

$$x_0 = 0; \quad P_0 = 0$$



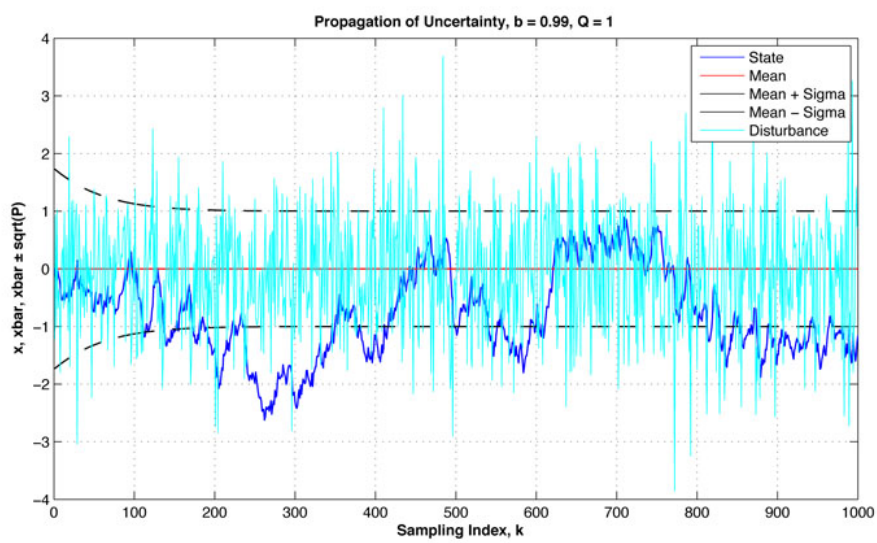
Example: Propagating a Scalar Probability Density Function

$$x_0 = 1; \quad P_0 = 0$$



Example: Propagating a Scalar Probability Density Function

$$x_0 = 0; \quad P_0 = 3$$



Gaussian PDF Propagation Program

```
%   Scalar Propagation of Mean and Variance
%   Rob Stengel
%   4/5/2011

    clear
    '=========================='
    'Propagation of Uncertainty'
    date
    b       =   0.99; b2      =   b*b;bb      =   1 - b^2;sqrb    =   sqrt(bb)
    w       =   []; x       =   []; xbar    =   [];
    x(1)    =   0;xbar(1) =   x(1)
    P(1)    =   3
    Q       =   1
    u       =   0
    for  k  =   1:999
        w(k)      =   randn(1);
        x(k+1)    =   b*x(k) + sqrb*u + sqrb*w(k);
        xbar(k+1) =   b*xbar(k) + sqrb*u;
        P(k+1)    =   b2*P(k) + bb*Q;
    end
    k       =   [1:1000];
    w(1000) =    randn(1);
    figure
    plot(k,x,'b',k,xbar,'r',k,(xbar+sqrt(P)),'k',k,(xbar-sqrt(P)),'k',k,w,'c'),grid
    title('Propagation of Uncertainty, b = 0.99, Q = 1'), xlabel('Sampling Index,
k'), ylabel('x, xbar, xbar ∓ sqrt(P)')
    legend('State','Mean','Mean + Sigma','Mean - Sigma','Disturbance')
```